Dynamics of the N-fold pendulum

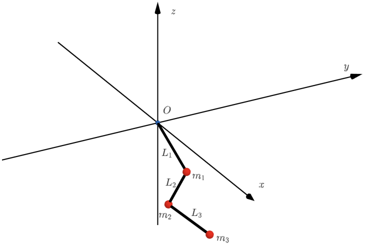
Lie group integrators approach to the N-fold pendulum

Part of a joint work with Elena Celledoni, Ergys Çokaj, Andrea Leone and Brynjulf Owren - "Lie Group integrators for mechanical systems." arXiv preprint

Overview of the presentation

In this talk, we go through the following points:

1. Some elements of the theory of Lie group integrators,
2. Key points in the derivation of the model for the N-fold pendulum,
3. Solving its equations of motion with Lie group integrators,
4. Some numerical experiments.



Basics of Lie group integrators

- They are used to solve differential equations whose solution evolves on a manifold M:

$$\dot{y}(t) = X|_{y(t)}, \;\; y(t_0) = y_0 \in M, \;\; X \in \mathfrak{X}(M).$$

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- Lie group integrators are based on some choices in the representation of the vector field:

  1. By means of the infinitesimal generator ψ_* of a transitive Lie group action $\psi : G \times M \to M$:

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3. By means of the machinery of connections.

Some key facts on Lie group actions

- **Definition:** Let M be a smooth manifold and $(G, \cdot)$ be a Lie group. The (left) action of the group G on M is a map $\psi : G \times M \to M$, $\psi(g, m) = \psi_g(m)$ such that:
  1. $\psi(1_G, m) = m \quad \forall m \in M$
  2. for any $g \in G$ the map $\psi_g : M \to M$ is a diffeomorphism and
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- The orbit of $m \in M$ is $\mathcal{O}(m) = \{\psi_g(m) : g \in G\} \subseteq M$.
- It is a transitive action if ψ_g is surjective for any $g \in M$.
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- Some relevant actions for our aims:
  1. $L_g(h) = g \cdot h$, (**left multiplication** - action of G onto itself)
  2. $R_g(h) = h \cdot g$, (**right multiplication** - action of G onto itself)
  3. $Ad_g(\xi) = L_g \circ R_{g^{-1}}(\xi) = g\xi g^{-1}$, (**adjoint action** - action of G onto $\mathfrak{g}$, $g \in G, \ \xi \in \mathfrak{g}$)

Two classes of Lie group integrators

- **Runge–Kutta–Munthe–Kaas (RKMK) methods**: The key idea is to transform locally the problem from M to the Lie algebra $\mathfrak{g}$ of a group G acting transitively on it:

  We solve this for one timestep Δt, then update y_n and repeat up to the final time T

$$\begin{cases} \sigma(0) = 0 \in \mathfrak{g}, \\ \dot{\sigma}(t) = dexp^{-1}_{\sigma(t)} \circ f \circ \psi(exp(\sigma(t)), y_n) \in T_{\sigma(t)}\mathfrak{g}, \quad n = 0, ..., N-1 \\ y(t) = \psi(exp(\sigma(t)), y_n) \in M. \end{cases}$$

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- **Commutator free Lie group methods**: The main idea is to update the position on the manifold by

$$y_{n+1} = \psi_{exp(\Delta t \sigma_1)} \circ ... \circ \psi_{exp(\Delta t \sigma_d)}(y_n)$$

  where the computation of the $\sigma_i \in \mathfrak{g}$ does not involve commutators.

Variable stepsize for RKMK methods

- One approach is based on the use of an embedded Runge-Kutta pair.
- In RKMK methods we are just updating on a manifold, while the real time integration, locally, happens on a linear space $\mathfrak{g}$.
- So variable stepsize in this context can be applied using an embedded RK pair on the local integration on $\mathfrak{g}$.
- If we adopt a RK pair of order $(p, \hat{p})$ and the one timestep approximations obtained with these RK methods on $\mathfrak{g}$ are $\sigma_1, \hat{\sigma}_1$, we can use $e_n = \|\sigma_1 - \hat{\sigma}_1\|$ as an estimate of the local truncation error.

Idea in the derivation of the vector field

- **Lagrangian of the system:** $L : (TS^2)^N \to \mathbb{R}$,

$$L(\boldsymbol{q}, \boldsymbol{\omega}) = T(\boldsymbol{q}, \boldsymbol{\omega}) - U(\boldsymbol{q}) =$$

$$= \frac{1}{2} \sum_{i,j=1}^{N} \Big(\sum_{k=max\{i,j\}}^{N} m_k \Big) L_i L_j \omega_i^T \hat{q}_i^T \hat{q}_j \omega_j - \sum_{i=1}^{N} \Big(\sum_{j=i}^{N} m_j \Big) g L_i e_3^T q_i,$$

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where $\boldsymbol{q} = (q_1, ..., q_N)$, $\boldsymbol{\omega} = (\omega_1, ..., \omega_N)$.

- For the case $N = 1$, we can consider the variations of the curve

$$q : [a, b] \to S^2$$

thanks to an $\varepsilon-$family of curves

$$q^{\varepsilon} : [a, b] \to S^2, \ q^{\varepsilon}(t) := exp\left(\varepsilon \widehat{\gamma(t)}\right) \cdot q(t),$$

and then, via Hamilton's principle, we can obtain the Euler-Lagrange equations on (TS^2), where $\gamma(t) \in T_{q(t)}S^2$ is an arbitrary curve with $\gamma(a) = \gamma(b) = 0$.

Vector field of the N-fold pendulum

- Repeating a similar idea, we can get to the system of ODEs for the N-fold spherical pendulum, where the $i-$th pair of 6 equations reads:

$$\dot{q}_i = \omega_i \times q_i,$$

$$(R(\boldsymbol{q})\dot{\omega})_i = \left[\sum_{\substack{j=1\\ j\neq i}}^{N} M(\boldsymbol{q})_{ij} |\omega_j|^2 \hat{q}_i q_j - \Big(\sum_{j=i}^{N} m_j \Big) g L_i \hat{q}_i e_3 \right].$$

- Here the symmetric block matrices $M(\boldsymbol{q})$ and $R(\boldsymbol{q})$ are so that

$$M(\boldsymbol{q})_{ii} = \Big(\sum_{j=i}^{N} m_j \Big) L_i^2 I_3, \quad M(\boldsymbol{q})_{ij} = \Big(\sum_{k=j}^{N} m_k \Big) L_i L_j I_3 = M(\boldsymbol{q})_{ji},\ i < j,$$

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where by A_{ij} we refer to the (i,j) 3×3 block matrix in A.

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$$Ad : SE(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3),$$

$$Ad((R, r), (u, v)) = (Ru, Rv + r \times Ru),$$

where we recall $Ad_g \xi := L_g \circ R_{g^{-1}}(\xi)$, $g \in SE(3)$, $\xi \in \mathfrak{se}(3)$.

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- Since $\mathfrak{se}(3) \simeq \mathbb{R}^6$, the Ad-action allows us to define the following Lie group action on $\mathbb{R}^6$

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- Infinitesimal generator of the action:

$$\psi_*((u, v))|_{(q,\omega)} = (u \times q, v \times q + u \times \omega)$$

Particular orbits of this action

- For a point $(q, \omega) \in \mathbb{R}^6$ such that $\omega^T q = 0$ and $(R, r) \in SE(3)$, we have that

$$(\bar{q}, \bar{\omega}) := \psi_{(R,r)}(q, \omega) = (Rq, R\omega + r \times Rq)$$

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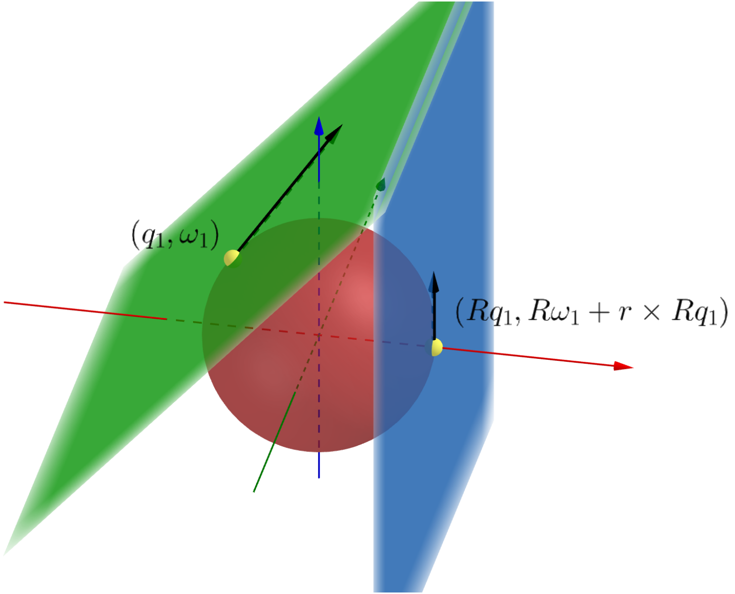
$$TS^2_r := \{(q, \omega) \in \mathbb{R}^6 : \ \omega^T q = 0, \ |q| = r\} \subset \mathbb{R}^6.$$

- This implies that

$$\mathcal{O}((q, \omega)) = \{m \in \mathbb{R}^6 : \ \exists g \in SE(3) \text{ s.t. } \psi_g(q, \omega) = m\} \subset TS^2_{||q||}$$

if $q^T \omega = 0$.

Visual representation of the action



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- More precisely, for any pair $m_1 = (q_1,\omega_1), m_2 = (q_2,\omega_2) \in TS^2$, there is (at least) a $g = (R, r) \in SE(3)$ such that

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- To verify that, just set

1. R such that $Rq_1 = q_2$,
2. $r = v - Ru \in \mathbb{R}^3$, where
   - $\omega_1 = u \times q_1$,
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denoted with $*$ the semidirect product of $SE(3)$ and with $\circ$ the group product of G, we set

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- We denote with $\boldsymbol{q} = (q_1, ..., q_N) \in (S^2)^N$ and $\boldsymbol{\omega} = (\omega_1, ..., \omega_N) \in T_{q_1}S^2 \times ... T_{q_N}S^2$.

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- **Infinitesimal generator**:

$$\psi_*(\xi)|_m = (u_1 \times q_1, u_1 \times \omega_1 + v_1 \times q_1, ..., u_N \times q_N, u_N \times \omega_N + v_N \times q_N),$$

where $\xi = [u_1, v_1, ..., u_N, v_N] \in \mathfrak{se}(3)^N$ and $m = (q_1, \omega_1, ..., q_N, \omega_N) \in (TS^2)^N$.

Representation via the infinitesimal generator

- We now have that

$$(R(\boldsymbol{q})\dot{\boldsymbol{\omega}})_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} M_{ij} |\omega_j|^2 \hat{q}_i q_j - \Big(\sum_{j=i}^{N} m_j \Big) g L_i \hat{q}_i e_3 = g_i(\boldsymbol{q}, \boldsymbol{\omega}) \in T_{q_i} S^2$$

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- So there is an $a_i : M \to \mathbb{R}^3$ such that $h_i(\boldsymbol{q},\boldsymbol{\omega}) = a_i(\boldsymbol{q},\boldsymbol{\omega}) \times q_i$, a possible choice is to set $a_i(\boldsymbol{q},\boldsymbol{\omega}) = q_i \times h_i(\boldsymbol{q},\boldsymbol{\omega})$.

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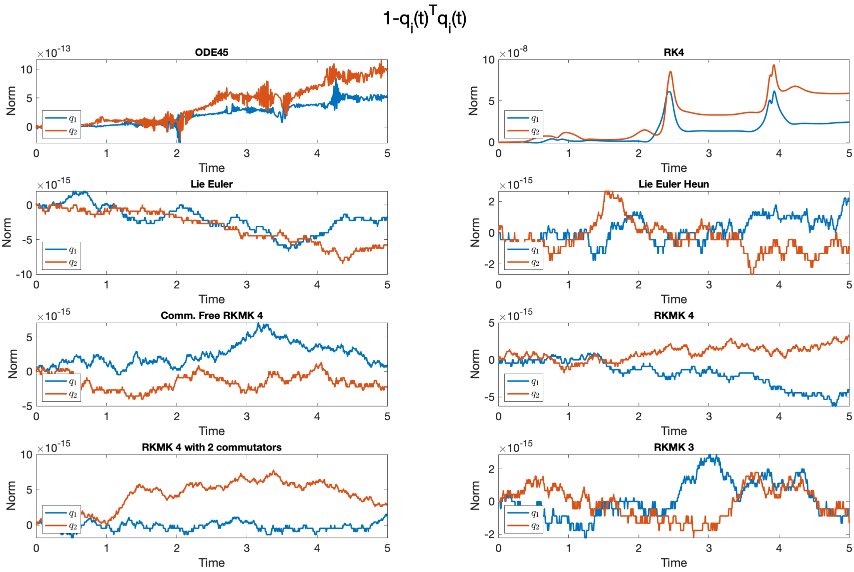
- Therefore, if we set $f : (TS^2)^N \to (\mathfrak{se}(3))^N$ as

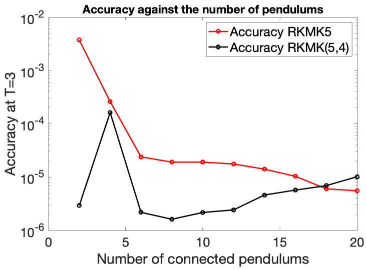
$$f(\boldsymbol{q},\boldsymbol{\omega}) = (\omega_1, a_1(\boldsymbol{q},\boldsymbol{\omega}), \ldots, \omega_N, a_N(\boldsymbol{q},\boldsymbol{\omega}))$$

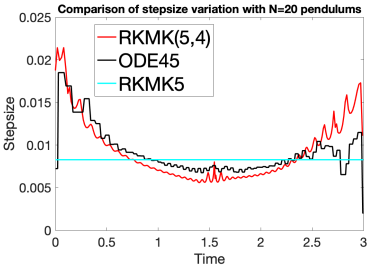
  follows $\psi_*(f(\boldsymbol{q},\boldsymbol{\omega}))|_{(\boldsymbol{q},\boldsymbol{\omega})} = F|_{(\boldsymbol{q},\boldsymbol{\omega})}$, where $F \in \mathfrak{X}(M)$ is the vector field defining the dynamics of the N-fold pendulum.

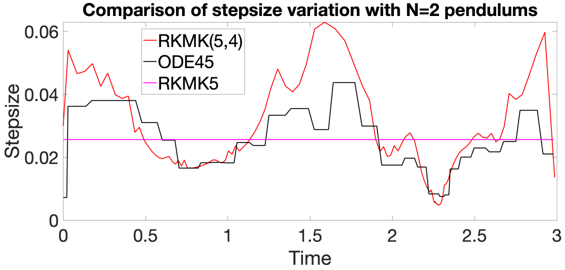
Preservation of the configuration manifold



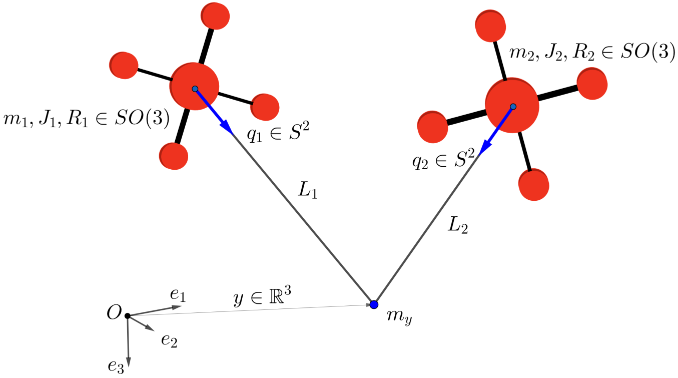
Experiments with variable stepsize







Another studied mechanical system



Configuration manifold: $Q = \mathbb{R}^3 \times (SO(3))^2 \times \left(S^2\right)^2$
Phase space: $M = TQ$
Group acting transitively: $\bar{G} = \mathbb{R}^6 \times (\mathrm{TSO}(3))^2 \times (SE(3))^2$

Thanks for the attention