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Integrable non-Hamiltonian systems: from B-integrability to Euler-Jacobi Theorem and back

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Introduction

The theory of integrability was born with Newtonian mechanics and developed rapidly and broadly. It finds its roots in Classical Mechanics, but it extends even to Quantum Mechanics. Mathematicians and physicists started being interested in finding differential equations with an explicit solution. After some initial result, as the explicit integration of Kepler's problem, they recognized the difficulty of solving non-trivial systems of differential equations. A key point in this realization was when, in 1841, Liouville discovered that the Riccati Equation can not be integrated by quadratures. From there on, the focus has shifted to finding conditions that guarantee the integrability by quadratures of systems of ODEs. The main approach was to explore theories and methods to determine if a dynamical system defined by a vector field X admits:

- first integrals (or integrals of motion), i.e. functions whose Lie derivative along X vanishes: $\mathcal{L}_X f = 0$, and/or
- dynamical symmetries (or symmetry fields), i.e. vector fields whose Lie derivative along X vanishes: $\mathcal{L}_X Y = 0$.

Two of the first classical results in this field involve these two invariants of the dynamics separately. Precisely, consider a smooth $n-$dimensional manifold M and a vector field $X \in \mathfrak{X}(M)$. The first mentioned result ensures that if X admits $(n-1)$ functionally independent first integrals $f_1, ..., f_{n-1} : M \to \mathbb{R}$, then it is integrable by quadratures. Moreover, it is important to recall even the approach of Lie to integrability, fully based on dynamical symmetries. Indeed, he discovered that when X admits $(n-1)$ linearly independent dynamical symmetries which pairwise commute, then it is integrable.

Starting from these initial results, the theory of integrability developed in various directions, with the introduction of more general approaches to the problem. We will present some of them in the following chapters. The research in the field of integrability began mainly in the area of Hamiltonian dynamics, with the classical Liouville-Arnold(-Mineur) Theorem. Then, it has expanded to dynamical systems defined on manifolds with a less rich geometric structure than the Poisson or symplectic structure typical of Hamiltonian systems (i.e. non-Hamiltonian).

In the Thesis, we follow the historical development of the research area and we start investigating the two main integrability theorems for Hamiltonian systems: Liouville-Arnold Theorem ([5]) and noncommutative integrability Theorem ([14] Nekhoroshev, [7] Mishchenko and Fomenko), analyzing the proof of the latter. In the proof, we follow a geometrical approach, working hence with symplectic geometry, with the isotropic fibration in invariant tori and the properties of bifibrations.

The analysis then extends in the Chapters 3, 4 and 5 to non-Hamiltonian dynamical systems. By "non-Hamiltonian" we do not mean that the results we are going to present do not apply to Hamiltonian systems, but just that they are not based on the properties of Hamiltonian vector fields. We precisely investigate the relation between two integrability results: Bogoyavlensky's integrability and Euler-Jacobi Theorem. The former ensures the fibration of the phase space in invariant tori and the possibility of conjugating the dynamics to a quasi-periodic flow on each fiber. On the other hand, Euler-Jacobi Theorem requires a time reparametrization before this conjugation and it is limited to the case in which the dynamics admits $n-2$ functionally independent first integrals, where n is the dimension of the manifold where the dynamics is defined. The key motivation behind the relevance of this second Theorem is that it ensures a sort of integrability notion for some mechanical systems which are not integrable in the sense of Bogoyavlensky.

We even prove these two Theorems. In particular, we give a complete proof of Euler-Jacobi Theorem (in Chapter 4) which is not easy to be found.

Proceeding in the analysis of non-Hamiltonian systems, we have been trying to answer to the following question: is there any sufficient condition for a system which is integrable in the sense of Euler-Jacobi to be even B-integrable? Chapter 5 is devoted to the analysis of this question and, in particular, here we recover three sufficient conditions which allow us to answer positively to the question above. Indeed, we focus on the case where the phase space M is a $2-$torus and X is a smooth non-vanishing vector field on M that is integrable à la Euler-Jacobi. Two of the three conditions mentioned above, allow to recover a linearly independent dynamical symmetry of X:

1. If there exists an $\alpha \in \Lambda^1(M)$ with $\mathcal{L}_X\alpha = 0$, then X admits a dynamical symmetry Y. Moreover, if α satisfies some additional property, we will prove that X is Bogoyavlensky's integrable too,
2. If on M there is a closed curve γ such that all the orbits of X starting there come back to it all at the same time, then the system is even Bogoyavlensky's integrable.

This generalization of the results presented for Hamiltonian systems (in Chapter 2), opens the doors to an even broader point of view on the theory of integrability. Indeed, the main tool we use to prove the integrability Theorems is to define a suitable change of coordinates which conjugates the dynamics to a quasi-periodic flow on the invariant level sets. Through the Thesis, we build this system of coordinates in the various settings, and we call them Liouville-coordinates in the case of non-Hamiltonian systems and action-angle variables in the more specific case of Hamiltonian systems.

Moreover, at the end of Chapter 3 we highlight a more conceptual approach to the problem of integrability. Indeed, we analyze again the proof of Bogoyavlensky's Theorem but, this time, in the light of torus actions. This section opens many interesting questions, one of these is: is there some torus action generating the fibration in $2-$tori appearing in Euler-Jacobi Theorem? We have just started working on this question which hence falls inside the "further work ideas".

To conclude this introduction, we briefly summarize the organization of the chapters of the Thesis as follows:

- Chapter 1: In the first Chapter we introduce the basic geometrical concepts which will be used in the following chapters.
- Chapter 2: This Chapter is dedicated to the description of the dynamical and geometrical properties of integrable and superintegrable Hamiltonian systems.
- Chapter 3: In the third Chapter we focus on Bogoyavlensky's Theorem. We state it, prove it, see an example and conclude focusing on torus actions.
- Chapter 4: This Chapter focuses on Euler-Jacobi Theorem. Precisely, here the aim is to give a detailed proof of the Theorem.
- Chapter 5: The last Chapter is fully devoted to the investigation of the relations between Bogoyavlensky's integrability and Euler-Jacobi theorem.

Chapter 1

Geometrical tools

1.1 Symplectic vector spaces

Systems of differential equations arising from classical mechanics and physics in general, have usually a phase space with a very rich geometrical structure. An important tool to describe these spaces (mainly in the case of Hamiltonian systems) is symplectic geometry.

Definition 1 (Symplectic vector space)**.** A symplectic vector space is a pair (E, Ω) where E is a vector space and $\Omega : E \times E \to \mathbb{R}$ is a non-degenerate skew-symmetric bilinear map.

Non-degeneracy means $dim\{w \in E : \Omega(v, w) = 0 \ \forall v \in E\} = 0$. Given a pair of symplectic vector spaces (E_1, Ω_1) and (E_2, Ω_2), a map $\varphi : E_1 \to E_2$ is a *symplectomorphism* if it is an isomorphism preserving the symplectic structure, i.e. $\varphi^*\Omega_2 = \Omega_1$.

Key example: The basic example of a symplectic vector space is $(\mathbb{R}^{2n}, \Omega_c)$ where, given the basis $\{e_1, ..., e_n, f_1, ..., f_n\}$,

- $\Omega_c(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \forall i, j = 1, ..., n,$

- $\Omega_c(e_i, f_j) = \delta_{ij}, \quad \forall i, j = 1, ..., n.$

This example suggests us to introduce the *symplectic matrix* $\mathbb{J}$ and to give an equivalent characterization of the symplectic $2-$form Ω_c. Let

$$\mathbb{J} = \begin{bmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{bmatrix},$$

then for any given pair of vectors $v, w \in \mathbb{R}^{2n}$, $\Omega_c(v, w) = v^T \mathbb{J}\, w$.

Definition 2 (Symplectic complement)**.** Let (E, Ω) be a symplectic vector space and $F \subset E$ one of its subspaces. The *symplectic complement* of F with respect to Ω is defined as

$$F^\Omega = \{w \in E \,:\, \Omega(v, w) = 0 \,\forall v \in F\}.$$

Sometimes we will call it the *symplectic orthogonal* of F.

Proposition 1. *Given a symplectic vector space* (E, Ω) *and a subspace* $F \subset E$*, then the two following results hold:*

- $dimE = dimF + dimF^{\Omega}$.
- $(F^{\Omega})^{\Omega} = F$

Proof. Define the inclusion map $I : F \to E$ and consider the pull-back

$$I^* : E^* \to F^*.$$

Recall that $\varphi \in E^*$ is a linear and continuous function $\varphi : E \to \mathbb{R}$, so the way I^* acts on φ is restricting it to F, namely $I^*\varphi = \varphi|_F : F \to \mathbb{R}$ is a linear and continuous map. Consider now the contraction map $i.\Omega : E \to E^*$ such that $i_v\Omega = \Omega(v, \cdot) : E \to \mathbb{R}$, which by non-degeneracy of Ω is an isomorphism. This gives that the map $\psi = I^* \circ i.\Omega : E \to F^*$ is composition of an isomorphism with a surjective map, so it is surjective. Recalling that $dim\, F^* = dim\, F$, follows the first result:

$$dimE = dim\, Ker\, \psi + dim\, Im\, \psi = dim\, Ker\, \psi + dim\, F.$$

The last remark to be done is that the kernel of ψ reads:

$$Ker\, \psi = \{v \in E : i_v\Omega(w) = 0\, \forall w \in F\} = F^{\Omega}.$$

To prove the second fact in the Proposition it is enough to check $(F^{\Omega})^{\Omega} \subset F$ since the other inclusion is trivial or to show they have the same dimension. We follow this second approach, relying on the first result we have just proven.

$$\begin{aligned} &dim\, F + dim\, F^{\Omega} = dimE, \\ &dim\, F^{\Omega} + dim\, (F^{\Omega})^{\Omega} = dimE, \text{ which when subtracted gives:} \\ &dim\, F - dim\, (F^{\Omega})^{\Omega} = 0. \end{aligned}$$

□

Definition 3 (Classification of subspaces)**.** Let (E, Ω) be a symplectic vector space and $F \subset E$ one of its subspaces.

- F is *isotropic* if Ω vanishes on it, namely if $F \subset F^{\Omega}$.
- F is *co-isotropic* if $F^{\Omega} \subset F$ (i.e. if F^{Ω} is isotropic).
- F is *Lagrangian* if both isotropic and co-isotropic, i.e. $F = F^{\Omega}$.
- F is *symplectic* if $F \cap F^{\Omega} = \{0\}$ (so $E = F \oplus F^{\Omega}$).

A key property of symplectic vector spaces (and even manifolds as we will see later) is that they are even-dimensional. This result is a direct consequence of the following proposition.

Proposition 2 (Existence of a Lagrangian subspace)**.** *Any symplectic vector space* (E, Ω) *admits a Lagrangian subspace.*

Proof. First of all let's notice that all symplectic vector spaces have at least the isotropic subspace $F = \{0\}$. Let $L \subset E$ be a maximal isotropic subspace. Namely, it is not strictly contained in any larger isotropic subspace. We now check that this is a Lagrangian subspace. Suppose by contradiction $L^\Omega \neq L$, which means $L \subset L^\Omega$ since it is at least isotropic. This means there exists $v \in L^\Omega \setminus L$. $\bar{L} = L \oplus \langle v \rangle$ is still an isotropic subspace of E by bilinearity of the symplectic form and it strictly contains L. This is a contradiction since L was supposed to be maximal. $\square$

This proposition has an immediate basic corollary.

Corollary 1. *Every symplectic vector space* (E, Ω) *is even-dimensional.*

Proof. Let $F \subset E$ be one of the Lagrangian subspaces of E. Then

$$dimE = dimF + dimF^\Omega = dimF + dimF = 2dimF$$

is even. $\square$

Definition 4 (Symplectic basis)**.** Consider a symplectic vector space (E, Ω). The basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ is symplectic if it gives the following representation of the symplectic form:

$$\Omega(v, w) = v^T \, \mathbb{J} \, w.$$

By Gram–Schmidt procedure, this basis can be constructed for any symplectic vector space.

Proposition 3. *Every symplectic vector space* (E, Ω) *of dimension* $2n$ *is symplectomorphic to* $(\mathbb{R}^{2n}, \Omega_c)$.

This implies that all symplectic vector spaces with the same dimension are symplectomorphic.

Proof. Let $\{e_1, ..., e_n, f_1, ..., f_n\}$ be the canonical basis of $(\mathbb{R}^{2n}, \Omega_c)$ and consider a symplectic basis $\{x_1, ..., x_n, y_1, ..., y_n\}$ for (E, Ω). We can define the map $\varphi : \mathbb{R}^{2n} \to E$ acting on the basis as follows:

$$e_i \to x_i,$$

$$f_i \to y_i.$$

This map is a diffeomorphism. Let's check it preserves the symplectic structure:

$$\begin{aligned}
(\varphi^*\Omega)(v, w) = \Omega(\varphi(v), \varphi(w)) = \Omega\Big(\sum_{i=1}^n v_i x_i + \sum_{i=1}^n v_{n+i} y_i, \sum_{j=1}^n w_j x_j + \sum_{j=1}^n w_{n+j} y_j\Big) = \\
= \sum_{i,j=1}^n v_i w_{n+j} \Omega(x_i, y_j) + \sum_{i,j=1}^n v_{n+i} w_j \Omega(y_i, x_j) = \\
= \sum_{i,j=1}^n v_i w_{n+j} \Omega_c(e_i, f_j) + \sum_{i,j=1}^n v_{n+i} w_j \Omega_c(f_i, e_j) = \Omega_c(v, w), \ \forall v, w \in \mathbb{R}^{2n}.
\end{aligned}$$

$\square$

1.2 Symplectic manifolds

Definition 5 (Symplectic manifold)**.** A pair (M, Ω) is a symplectic manifold if M is a smooth manifold and $\Omega \in \Lambda^2(M)$ is a non-degenerate and closed 2-form.

Non-degenerancy of Ω means precisely that for any $x \in M$ the pair (T_xM, Ω_x) is a symplectic vector space. Since $dimM = dim\, T_xM$, then every symplectic manifold is even-dimensional.

Theorem 1 (Darboux)**.** *Let (M, Ω) be a symplectic manifold. For every point $m \in M$ there exists a chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ at m such that*

$$\Omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

These coordinates will be called *Darboux coordinates.* This Theorem can be reinforced into the so-called Carathéodory-Jacobi-Lie Theorem.

Theorem 2 (Carathéodory-Jacobi-Lie)**.** *Let (M, Ω) be a symplectic manifold of dimension $2n$. Suppose in a neighbourhood of a point $m \in M$ there exists a family of k-differentiable functions ($k \leq n$) pairwise in involution $F_1, ..., F_k$, with linearly independent differentials. Then they can be completed to a system of Darboux coordinates in a neighbourhood $U \subset M$ of m. Namely, there exist $2n - k$ functions $\{T_1, ..., T_k, P_1, ..., P_{n-k}, Q_1, ..., Q_{n-k}\}$ such that*

$$\Omega|_U = \sum_{i=1}^{k} dF_i \wedge dT_i + \sum_{j=1}^{n-k} dP_j \wedge dQ_j.$$

The proof of this Theorem can be seen for example in [10]. Symplectic manifolds emerge naturally from classical mechanics and in particular from Hamiltonian dynamics. Let's define a basic symplectic manifold: *the phase space.*

The phase space and Hamiltonian vector fields

Let Q be a smooth manifold, which can be seen as the configuration manifold of a mechanical system. We denote with T^*Q its cotangent bundle and with $\pi : T^*Q \to Q$ the canonical projection. T^*Q is a manifold and it is called the *phase space* of the system. This manifold is naturally endowed with a symplectic structure. We can define on it a canonical symplectic two form starting from the so called *tautological one form* $\theta \in \Lambda^1(T^*Q)$. Let n be the dimension of Q and consider a point $q = (q_1, ..., q_n) \in Q$. A point in the $2n-$dimensional manifold T^*Q is of the form $(q, p) = (q_1, ..., q_n, p_1, ..., p_n)$.

In local coordinates the tautological one form can be written as

$$\theta = \sum_{i=1}^{n} p_i dq^i.$$

It can be equivalently characterized with a coordinate free approach as follows:

- Let $\pi : M = T^*Q \to Q$, $\pi(q, p) = q$ be the canonical projection map.

- Let $m = (q, p) \in M$ be a point of the cotangent bundle. If its base point is $q = \pi(m)$, then p is a linear mapping $p : T_qQ \to \mathbb{R}$.

- The pull-back of the canonical projection at q is the linear map

$$\pi^* : T_q^*Q \to T_m^*M.$$

- We define the tautological one form at $m = (q, p)$ as the differential form $\theta \in \Lambda^1(M)$ such that $\theta_m = \pi^* p$ and hence

$$\theta_m(Y) = p(\pi_* Y), \quad \forall Y \in T_m M,$$

where $\pi_* : T_m M \to T_q Q$ is the push-forward of π at m.

This one form naturally generates a symplectic form on T^*Q, which in local coordinates reads:

$$\Omega = -d\theta = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

This is the *canonical symplectic form* of the phase space. The fact that it is a symplectic form can be seen by checking that the representative matrix of this skew-symmetric mapping is exactly the symplectic matrix. Indeed,

- $\Omega(\partial_{q_j}, \partial_{p_k}) = \sum_{i=1}^{n} dp_i(\partial_{p_k}) dq_i(\partial_{q_j}) = \delta_{jk}$,

- $\Omega(\partial_{q_j}, \partial_{q_k}) = \Omega(\partial_{p_j}, \partial_{p_k}) = 0$.

Symplectic manifolds can be seen as the generalization of the phase space and the construction done above explains the strong relation between classical mechanics, dynamical systems and symplectic geometry.

Definition 6 (Hamiltonian vector field)**.** Given a symplectic manifold (M, Ω) and a smooth function $H : M \to \mathbb{R}$, the Hamiltonian vector field of H is defined by

$$i_{X_H}\Omega = -dH,$$

where $i_{X_H} : \Omega^2(M) \to \Omega^1(M)$ is the contraction map along the vector field X_H.

Due to the presence of invariant differential forms, we will often apply Cartan's magic formula, so let's introduce it and prove it here for two types of differential forms.

Proposition 4 (Cartan's formula)**.** *Given a smooth manifold M and a vector field $X \in \mathfrak{X}(M)$, then the following identity holds*

$$\mathcal{L}_X \alpha = d i_X \alpha + i_X d\alpha \tag{1.1}$$

for every $k-$form α on M.

Proposition 5. *If equation (1.1) holds for $1-$forms, then it holds even for closed $2-forms$.*

Proof. All the elements involved in the identity (1.1) are considered locally, so we can suppose the closed $2-$form Ω to be exact, since it is true at least locally. Let $\Omega = d\alpha$ for some $\alpha \in \Lambda^1(M)$. Since $d\Omega = 0$, identity (1.1) in this case reduces to

$$\mathcal{L}_X\Omega = di_X\Omega.$$

We conclude the proof with the following direct computation

$$\mathcal{L}_X\Omega = \mathcal{L}_X d\alpha = d\mathcal{L}_X\alpha = d\Big(di_X\alpha + i_X d\alpha\Big) = d(i_X(d\alpha)) = di_X\Omega.$$

□

Proposition 6. *Cartan's formula holds for* $1-$*forms.*

Proof. Considering a local coordinate chart $(U, x_1, ..., x_n)$, we can express the $1-$form α as:

$$\alpha = \sum_{i=1}^{n} f_i dx^i \in \Lambda^1(M), \text{ with } f_1, ..., f_n \in \mathcal{C}^\infty(M).$$

By linearity of all the involved operators, we can reduce the proof to the form fdx^1. In this local chart the field along which we are contracting takes the form $X = \sum_{k=1}^{n} X^k \partial_{x_k}$. By direct computations we have

$$\begin{aligned} L_X(fdx^1) &= L_X(f)dx^1 + fL_X(dx^1) = X(f)dx^1 + fdL_X(x^1) = \\ &= X(f)dx^1 + fd(X(x_1)) = df(X)dx^1 + fdX^1, \end{aligned}$$

$$\begin{aligned} di_X(fdx^1) &= d(fdx^1(X)) = d(fX^1) = X^1df + fdX^1, \\ i_X(d(fdx^1)) &= i_X(df \wedge dx^1) = df(X)dx^1 - dx^1(X)df = \\ &= df(X)dx^1 - X^1df, \end{aligned}$$

which allows to conclude explicitly summing these terms as in Cartan's formula.

□

As a natural consequence of Cartan's magic formula, we see that the symplectic form is invariant for a Hamiltonian vector field:

$$\mathcal{L}_{X_H}\Omega = di_{X_H}\Omega + i_{X_H}d\Omega = -d^2H = 0.$$

Moreover, the Hamiltonian H is a first integral of X_H:

$$\mathcal{L}_{X_H}(H) = dH(X_H) = \Omega(X_H, X_H) = 0,$$

where we have concluded by the skew-symmetry of Ω.

To prove noncommutative integrability, we will use various concepts of differential geometry, so first of all we need to define them here.

Definition 7 (Lagrangian, isotropic, co-isotropic submanifolds)**.** Let (M, Ω) be a symplectic manifold and $N \subset M$ be a submanifold. N is said to be Lagrangian (resp. isotropic, co-isotropic) if for any $n \in N$ the tangent space T_nN is Lagrangian (resp. isotropic, co-isotropic), where its symplectic orthogonal is defined as

$$(T_nN)^\Omega = \{w \in T_nM \,:\, \Omega(v, w) = 0\,\forall v \in T_nN\}.$$

Another basic concept we will use is the one of tangent distribution.

Definition 8 ((Tangent) distribution)**.** Let M be a smooth $n-$dimensional manifold. A tangent distribution $\mathcal{D}$ of constant rank k, with $0 \leq k \leq n$, is a collection of $k-$dimensional tangent spaces defined as follows:

$$\mathcal{D} = \bigcup_{x \in M} V_x$$

where for any $x \in M$ the vector space $V_x \subset T_x M$ has dimension k.

Similarly to what happens with integral curves of vector fields (which are $1-$rank distributions), for a $k-$dimensional distribution $\mathcal{D}$ it is reasonable to ask if there is a $k-$dimensional submanifold $N \subset M$ whose tangent spaces are pointwise coinciding with the elements $\mathcal{D}_x = V_x$ of $\mathcal{D}$.

Theorem 3 (Frobenius)**.** *A distribution $\mathcal{D}$ on the manifold M is (completely) integrable if and only if it is closed with respect to the Lie bracket, namely*

$$\forall X, Y \in \mathcal{D},\ [X, Y] \in \mathcal{D}.$$

This condition guaranteeing the integrability of the distribution is called involutivity. We will not prove this Theorem here since it is out of our purposes, but a proof of it can be found in [6]. Let's present an example of a non-integrable distribution.
Consider the two vector fields $X = \partial_x$, $Y = x\partial_z + \partial_y$ on $M = \mathbb{R}^3$ and define the rank-2 distribution $\mathcal{D} = span\{X, Y\}$.

$$[X, Y] = \partial_z \notin \mathcal{D},$$

so we conclude by Frobenius Theorem that it is not integrable. This means there is no $2-$dimensional submanifold $N \subset M$ to which $\mathcal{D}$ is pointwise tangent. We will see that integrability of tangent distributions is related to integrability of dynamical systems.

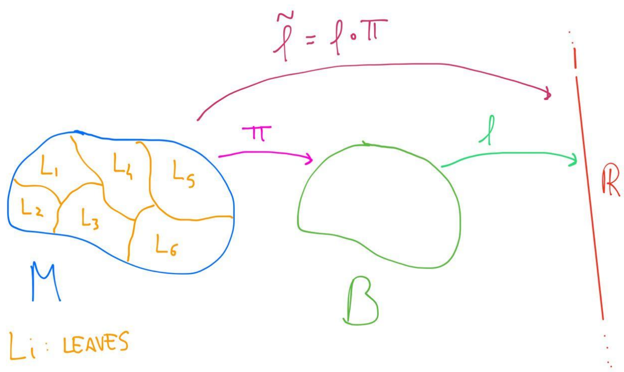
Let (M, Ω) be a symplectic manifold and $\mathcal{F}$ a foliation of M. The foliation is said to be *Lagrangian* (resp. *isotropic*, *co-isotropic*) if all its leaves are *Lagrangian* (resp. *isotropic*, *co-isotropic*) submanifolds. Moreover, given a foliation $\mathcal{F}$ we can define its polar $\mathcal{F}^{\perp}$, if it exists, as the foliation whose leaves have as tangent spaces the symplectic orthogonal of the tangent spaces to the leaves of $\mathcal{F}$, namely:

$$\forall N \text{ leaf of } \mathcal{F}^{\perp} \forall x \in N,\ T_x N = (T_x S)^{\Omega}$$

where S is the leaf of the foliation $\mathcal{F}$ where x lives. If the polar does exist, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is called a *dual-pair* or a *bifoliation.* Moreover, when $\mathcal{F}$ is isotropic (resp. co-isotropic, Lagrangian) then the polar $\mathcal{F}^{\perp}$, if it exists, is co-isotropic (resp. isotropic, Lagrangian). We define *dimension of a foliation* to be the dimension of its leaves and in case $\mathcal{F}$ is isotropic and $dim\, M = 2n$, then follows $dim\mathcal{F} \leq n$, while if it is co-isotropic then $dim\mathcal{F} \geq n$. Suppose that the dual pair $(\mathcal{F}, \mathcal{F}^{\perp})$ is well defined and that each of these two foliations are defined by a submersion $\pi_1 : M \to B_1$ and $\pi_2 : M \to B_2$ which are fibrations, then such a pair is even called *bifibration.* Given a foliation $\mathcal{F}$, we can denote with F the $k-$dimensional distribution composed by the collection of tangent spaces to the leaves of $\mathcal{F}$. We can define the distribution $F^{\perp}$ which associates to any point $x \in M$ the symplectic orthogonal to the respective element in $F : F_x$. Hence the existence of the polar $\mathcal{F}^{\perp}$ is equivalent to the integrability of the orthogonal distribution $F^{\perp}$.

Definition 9 (First integral of a foliation)**.** A first integral f of a foliation $\mathcal{F}$ is a function which is constant on its leaves. If the foliation is defined by the submersion $\pi : M \to B$, then the first integrals are the lifts via π of all the functions defined on B:

$$\tilde{f} = f \circ \pi : M \to \mathbb{R}.$$



This definition is coherent with the classical definition of first integrals of a vector field. Indeed, let $X \in \mathfrak{X}(M)$ and $f : M \to \mathbb{R}$ a first integral of X, i.e. $\mathcal{L}_X f = 0$. Then the function f is constant on the orbits of X which are leaves of the foliation defined by X on M.

Proposition 7 (Characterization of first integrals)**.** *A function f is a first integral of a foliation $\mathcal{F}$ if and only if its Hamiltonian vector field lives in $F^{\perp}$.*

Proof. A first integral is a function which is constant on the leaves of the foliation, so these leaves are contained in the level sets of f. Moreover we know that for any $Y \in \mathfrak{X}(M)$, $\mathcal{L}_Y f = df(Y) = \Omega(Y, X_f)$. By definition, being constant on each leaf means that the function f does not vary along any vector field $Y \in F$, which is hence equivalent to say

$$\forall Y \in F, \Omega(X_f, Y) = 0, \text{ i.e. } X_f \in F^{\perp}$$

□

Proposition 8. *Let (M, Ω) be a symplectic manifold and $f, g \in \mathcal{C}^{\infty}(M)$. Then*

$$[X_f, X_g] = X_{\{f,g\}}.$$

Proof. The proof can be done with the following direct computation for any $h \in \mathcal{C}^{\infty}(M)$:

$$\begin{aligned}[X_f, X_g](h) = X_f(X_g(h)) - X_g(X_f(h)) = \{f, \{g, h\}\} - \{g, \{f, h\}\} = \\ = \{h, \{g, f\}\} = -\{\{g, f\}, h\} = \{\{f, g\}, h\} = X_{\{f,g\}}(h).\end{aligned}$$

□

As an immediate consequence, denoted with $\mathfrak{X}_{Ham}(M)$ the vector space of Hamiltonian vector fields on M, then:

$$[\mathfrak{X}_{Ham}(M), \mathfrak{X}_{Ham}(M)] \subset \mathfrak{X}_{Ham}(M).$$

Proposition 9 (Existence of a polar)**.** *Let $\mathcal{F}$ be a foliation of a symplectic manifold (M, Ω). Then $\mathcal{F}$ admits a polar if and only if the Poisson bracket of every two first integrals of $\mathcal{F}$ is again a first integral.*

Proof. The polar does exist if and only if the distribution $F^\perp$ is integrable, i.e. by Frobenius Theorem it is closed with respect to the Lie Bracket.

Step 1: Let's assume that $\mathcal{F}$ admits a polar $\mathcal{F}^\perp$. This means that $F^\perp$ is an integrable distribution. Hence for any pair of first integrals $f, g \in \mathcal{C}^\infty(M)$ we have $[X_f, X_g] \in F^\perp$. This allows us to conclude, since

$$[X_f, X_g] = X_{\{f,g\}} \in F^\perp$$

means that $\{f, g\}$ is a first integral too.

Step 2: Let's now assume for any pair of first integrals $f, g \in \mathcal{C}^\infty(M)$, $\{f, g\}$ is again a first integral. To show that $F^\perp$ is completely integrable and hence $\mathcal{F}^\perp$ does exist, we rely again on Frobenius Theorem. By assumption, we know that $[X_f, X_g] = X_{\{f,g\}} \in F^\perp$. We conclude by noticing that the distribution has local bases made by the Hamiltonian vector fields of first integrals of $\mathcal{F}$ and the result follows by bilinearity of the Lie bracket. $\square$

Indeed, any point $m \in M$ has a neighbourhood U where there are

$$c = 2n - dim(\mathcal{F})$$

functions $f_1, ..., f_c$ which have everywhere independent differentials and are first integrals of $\mathcal{F}$. For instance, it is enough to take these local coordinates to be transversal to the leaf where m lives. $X_{f_1}, ..., X_{f_c}$ belong to $F^\perp$ and being linearly independent they form a local basis.

Assuming the existence of the polar $\mathcal{F}^\perp$ of $\mathcal{F}$, the following two corollaries hold.

Corollary 2. *The leaves of $\mathcal{F}$ are generated by the flows of the Hamiltonian vector fields of the first integrals of $\mathcal{F}^\perp$.*

Corollary 3. *A function is a first integral of $\mathcal{F}$ if and only if it is in involution with every first integral of $\mathcal{F}^\perp$.*

1.3 Poisson Manifolds

Definition 10 (Poisson manifold)**.** A Poisson manifold is a pair $(M, \{\cdot, \cdot\})$ where M is a manifold and $\{\cdot, \cdot\}$ is called a *Poisson bracket*. The map

$$\{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) :$$

- is bilinear and skew-symmetric,

- satisfies Leibniz's rule, namely

$$\{fg,h\} = f\{g,h\} + \{f,h\}g\ \forall f,g,h \in \mathcal{C}^\infty(M),$$

- satisfies Jacobi's identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0, \quad \forall f,g,h \in \mathcal{C}^\infty(M).$$

The notion of Hamiltonian vector field is well defined even on a Poisson manifold. The Hamiltonian vector field X_h of $h \in \mathcal{C}^\infty(M)$ reads

$$X_h(f) = \{h,f\}, \quad \forall f \in \mathcal{C}^\infty(M).$$

This implies that a function $f \in \mathcal{C}^\infty(M)$ is a first integral of X_h if and only if h and f are in involution, i.e. $\{h,f\} = 0$.

Definition 11 (Poisson mapping)**.** A smooth map $\varphi : M_1 \to M_2$ between two Poisson manifolds $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$ is called a Poisson mapping if it preserves the Poisson structure, namely

$$\{f,g\}_2 \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_1, \quad \forall f,g \in \mathcal{C}^\infty(M_2).$$

Just to clarify, this means $\varphi^*\{f,g\}_2 = \{\varphi^* f, \varphi^* g\}_1$. A Poisson manifold is naturally foliated into leaves which have a symplectic structure and are called *symplectic leaves*. In order to present this symplectic structure, for every $x \in M$ we can define the subspace $S_x \subset T_xM$

$$S_x = \{v \in T_xM : X_f(x) = v \text{ for some } f \in \mathcal{C}^\infty(M)\}.$$

The collection of all the subspaces S_x, $x \in M$, is a completely integrable distribution and so it defines a foliation $\mathcal{F}$ of M, whose leaves are its integral manifolds. Moreover, these leaves are endowed with a symplectic structure which is naturally defined as $\Omega(v,w) := \{f,g\}$, where $f,g \in \mathcal{C}^\infty(M)$ are the unique functions such that $X_f(x) = v$ and $X_g(x) = w$. The simplest case of Poisson manifold is the one of a connected symplectic manifold, in this case we have a single symplectic leaf covering the whole manifold.

An alternative way to define a Poisson manifold is as a pair (M,Π) where M is a smooth $n-$dimensional manifold and Π is a bi-vector field satisfying the relation $[\Pi,\Pi]^{ST} = 0$ (where this is the Schouten bracket). This bi-vector field is called *Poisson structure* for the manifold M. For any pair of functions $f,g \in C^\infty(M)$ we can set

$$\{f,g\} = (df \wedge dg)(\Pi).$$

For any pair $U = X_1 \wedge Y_1$ and $V = X_2 \wedge Y_2$ of bi-vector fields on M,

$$\begin{aligned}[U,V]^{ST} &= [U, X_2 \wedge Y_2]^{ST} = [U, X_2]^{ST} \wedge Y_2 + (-1)^{(2-1)1} X_2 \wedge [U, Y_2]^{ST} = \\ &= X_2 \wedge [Y_2, X_1 \wedge Y_1]^{ST} - [X_2, X_1 \wedge Y_1]^{ST} \wedge Y_2 = X_2 \wedge X_1 \wedge [Y_2, Y_1] + ... \\ &... + X_2 \wedge [Y_2, X_1] \wedge Y_1 - X_1 \wedge [X_2, Y_1] \wedge Y_2 - [X_2, X_1] \wedge Y_1 \wedge Y_2.\end{aligned}$$

These computations follow from the fact that the Schouten bracket of two vector fields coincides with their Jacobi-Lie bracket and, when it is computed between a bi-vector field and a vector field, it is skew-symmetric. Consider a $2n-$dimensional smooth manifold M locally coordinatized by

$$\{x_1, ..., x_n, y_1, ..., y_n\}.$$

The standard Poisson structure in this chart reads:

$$\Pi_s = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

The Poisson bracket generated by this structure is the classical one:

$$\begin{aligned}\{f, g\} &= (df \wedge dg)\Big(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}\Big) = \sum_{i=1}^{n}(df \wedge dg)\Big(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}\Big) = \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}.\end{aligned}$$

Given a smooth function $h \in \mathcal{C}^\infty(M)$, its Hamiltonian vector field with respect to the Poisson structure Π is defined as

$$X_h := i_{-dh}\Pi.$$

This definition is compatible with the classical definition of Hamiltonian vector fields when we consider the standard structure Π_s:

$$\begin{aligned}i_{-dh}\Pi_s &= \sum_{i=1}^{n} \Big(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}\Big)(-dh) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(-dh)\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_i}(-dh)\frac{\partial}{\partial x_i} = \\ &= \sum_{i=1}^{n} -\frac{\partial h}{\partial x_i}\frac{\partial}{\partial y_i} + \frac{\partial h}{\partial y_i}\frac{\partial}{\partial x_i} = X_h.\end{aligned}$$

Definition 12 (Casimir)**.** Given a Poisson manifold $(M, \{\cdot,\cdot\})$, a Casimir for M is a function $f \in \mathcal{C}^\infty(M)$, which is in involution with all other functions, i.e. $\{f, g\} = 0$ for all $g \in \mathcal{C}^\infty(M)$. Equivalently, f is constant along the flow of any Hamiltonian vector field.

For connected symplectic manifolds, Casimirs are constant functions. Indeed, $X_C = 0$ implies $dC = 0$ and hence they are constant. When the Poisson structure is degenerate, for example if we consider the following Poisson structure on $\mathbb{R}^4$

$$\Pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

then there are non-constant Casimir functions. In this case for example the function $f(x, y, z, t) = z$ is a Casimir, since $i_{-dz}\Pi = 0$, and it is not constant.

Proposition 10. *Let (M, Ω) be a symplectic manifold and $\pi : M \to B$ a surjective submersion with connected level sets. Denote with $\mathcal{F}$ the foliation of M whose leaves are the fibers of π. $\mathcal{F}$ has a polar foliation $\mathcal{F}^\perp$ if and only if there exists a Poisson structure on B compatible to the one of M, i.e. π becomes a Poisson mapping.*

Proof. Assume there is a polar $\mathcal{F}^\perp$. Given an open subset $U \subset B$ and a pair of functions $f, g \in \mathcal{C}^\infty(U)$, then their lift via π are first integrals of $\mathcal{F}$ in $\pi^{-1}(U)$. By the closure of the set of first integrals with respect to the Poisson bracket, even $\{f \circ \pi, g \circ \pi\}_M$ is a first integral. Moreover, since the level sets of π are connected, there exists a function $\mathcal{P}_{f,g}$ defined in U such that

$$\{f \circ \pi, g \circ \pi\}_M = \mathcal{P}_{f,g} \circ \pi,$$

which means that this Poisson bracket just depends on the fiber of π. So we can just set $\{f, g\}_B := \mathcal{P}_{f,g}$.

Conversely, assume there exists a compatible Poisson structure $\{\cdot, \cdot\}_B$ on B. Since any first integral of $\mathcal{F}$ is the lift to M via π of a function $f \in \mathcal{C}^\infty(B)$, then the Poisson bracket of every two first integrals of F is still a first integral, which implies that $\mathcal{F}$ admits a polar. In fact

$$\{f \circ \pi, g \circ \pi\}_M = \{f, g\}_B \circ \pi$$

is a first integral. This implies that the set of first integrals of $\mathcal{F}$ is closed with respect to the Poisson structure of M and hence $\mathcal{F}^\perp$ does exist.

□

Proposition 11. *Consider the same setting of the previous Proposition. Assuming the polar $\mathcal{F}^\perp$ does exist, then the following conditions are equivalent:*

1. *The leaves of $\mathcal{F}$ are isotropic,*
2. *The number of independent local Casimirs of B coincides with the dimension of the level sets of π,*
3. *The leaves of $\mathcal{F}$ are generated by the Hamiltonian vector fields of the lifts to M of the Casimirs of B,*
4. *The first integrals of $\mathcal{F}^\perp$ are exactly the lifts to M of the Casimirs of B.*

Proof. We see that (3) and (4) are just a restatement of the same condition since the Hamiltonian vector fields of the first integrals of $\mathcal{F}^\perp$ generate the leaves of $\mathcal{F}$. Moreover, these 2 conditions are a restatement even of (2). Indeed, since the Casimirs are functionally independent, so are their lifts. Hence their number coincides with the dimension of the leaves of $\mathcal{F}$ if and only if they are generated by the Hamiltonian vector fields of these lifts.

It now remains to prove that (1) and (2) are equivalent. The number n_x of independent Casimirs of B at $\pi(x)$ coincides with the number of independent first integrals at x both of $\mathcal{F}$ and $\mathcal{F}^\perp$. The Hamiltonian vector fields of these functions are both tangent to the leaves of $\mathcal{F}$ and $\mathcal{F}^\perp$, then

$$n_x = dim(F_x \cap F_x^\perp).$$

But $dim(F_x \cap F_x^\perp) \leq dim(F_x)$, which implies that $n_x = dim(F_x)$ if and only if $dim(F_x) = dim(F_x \cap F_x^\perp)$, i.e. $\mathcal{F}$ is isotropic.

□

Before going on, let's remark that condition (2) gives important information about the rank of the Poisson structure induced on the base manifold B. Indeed, the dimension of the leaves of $\mathcal{F}$ coincides with $dim\, M - dim\, B$. On the other hand, the rank of the Poisson structure at $\pi(x)$, with $x \in M$, is equal to $dim\, B - n_x$, where n_x is the number of independent Casimirs of B at $\pi(x)$. In particular, the Poisson structure is non-degenerate if and only if it just has constant functions as Casimirs. We conclude noticing that n_x coincides with the dimension of the level sets of π; hence follows that the rank of the induced Poisson structure on B is

$$dim\, B - (dim\, M - dim\, B) = 2dim\, B - dim\, M.$$

Proposition 12. *Let (M, Ω) be a symplectic manifold, $\pi_1 : M \to B$ a surjective submersion whose level sets are connected and isotropic. Let $\mathcal{F}$ be the foliation with leaves coinciding with the fibers of π_1. Suppose it has a polar $\mathcal{F}^\perp$ with leaves coinciding with the connected components of the level sets of the surjective submersion $\pi_2 : M \to A$. Both A and B are manifolds. Then there exists a unique surjective submersion $\pi_3 : B \to A$ such that $\pi_3 \circ \pi_1 = \pi_2$. Moreover, the symplectic leaves of B are the connected components of the level sets of π_3.*

Proof. The foliation $\mathcal{F}$ is isotropic, so the leaves of $\mathcal{F}^\perp$ are co-isotropic submanifolds. Moreover, we know that these leaves contain the leaves of the foliation $\mathcal{F}$ by construction, hence they are a disjoint union of level sets of π_1. This implies that a map of the form $\pi_3 : B \to A$ is well defined. π_3 is a surjective submersion since both π_1 and π_2 have these properties.

To show the second part of the statement, we start noticing that the dimension of the symplectic leaves coincides with the one of the level sets of π_3. Indeed, the one of the symplectic leaves is equal to the rank of the Poisson structure on B, precisely it coincides with $2\, dim\, B - dim\, M$. Moreover, the dimension of A is equal to the one of the leaves of $\mathcal{F}$ and hence $dim\, A = dim\, M - dim\, B$. This implies that

$$dim\, \pi_3^{-1}(\{c\}) = dim\, B - dim\, A = dim\, B - (dim\, M - dim\, B) = 2\, dim\, B - dim\, M.$$

Moreover, the tangent spaces to the leaves and to the connected components of the level sets of π_3 coincide (it can be seen by lifting a system of local coordinates from A to B), so we can conclude the desired result.

$\square$

Chapter 2

Superintegrability of Hamiltonian systems

In this Chapter we analyze two classical results in the field of integrability of vector fields. They are Liouville-Arnold Theorem and Mishchenko and Fomenko Theorem for superintegrable systems. We will consider Hamiltonian vector fields defined on a symplectic manifold (M, Ω).

2.1 Completely integrable systems

The notion of complete integrability is related to the Liouville-Arnold Theorem.

Definition 13 (Regular level set)**.** Consider a manifold M and a submersion $F = (f_1, ..., f_k) : M \to \mathbb{R}^k$. A level set $N = \{m \in M : F(m) = \bar{f}\}$ is said to be regular if

$$df_1 \wedge ... \wedge df_k \neq 0 \text{ on } N.$$

Theorem 4 (Liouville-Arnold)**.** *Consider a $2n-$dimensional symplectic manifold (M, Ω) and a Hamiltonian vector field X_H of Hamiltonian function $H \in \mathcal{C}^\infty(M)$. Let*

$$F = (f_1, ..., f_n) : M \to \mathbb{R}^n$$

be a submersion whose components are functionally independent first integrals of X_H (i.e. $\{H, f_i\} = 0$ for any $i = 1, ..., k$) which are pairwise in involution. We additionally assume that the level sets of F are regular, compact and connected. Then each level set is diffeomorphic to a $n-$dimensional torus $\mathbb{T}^n$. Furthermore, for each level set $N = \{F = \bar{f}\}$ there exists a tubular neighbourhood $\mathcal{U}(N)$ endowed with a system of Darboux coordinates $(a_1, ..., a_n, \alpha_1, ..., \alpha_n)$, with $a_i = a_i(F)$ for any $i = 1, ..., n$. Precisely, $\mathcal{U}(N) \simeq \mathcal{F} \times \mathbb{T}^n$ with $\mathcal{F} \subseteq \mathbb{R}^n$ an open neighbourhood of $\bar{f}$ and the vector field X_H restricted to $\mathcal{U}(N)$ reads:

$$X|_{\mathcal{U}(N)} = \sum_{i=1}^{n} \omega_i(a_1, ..., a_n) \frac{\partial}{\partial \alpha_i}, \quad \text{where } \omega_i : \mathcal{F} \to \mathbb{R} \text{ are differentiable functions.}$$

This result is not explicitly proven here, but the proof can be found in [1] or [5]. Indeed, we can see this Theorem as a particular case of a noncommutative integrability Theorem for Hamiltonian vector fields, which will be proven in the remaining part of the Chapter.

An immediate consequence of Liouville-Arnold Theorem is that Hamiltonian dynamical systems with a $2-$dimensional phase space are completely integrable, while when the dimension is 4, it is enough to have a single first integral f functionally independent from the Hamiltonian function H. Precisely, since

$$0 = \mathcal{L}_{X_H} f = X_H(f) = \{H, f\},$$

the involutivity comes for free due to the Hamiltonian nature of the field X_H. We now analyze the case of two uncoupled harmonic oscillators. This example allows to recover the differences between the description of the dynamics provided by Liouville-Arnold Theorem and the noncommutative approach to integrability.

2.1.1 Two uncoupled oscillators

Let

$$H = \omega_1 \frac{p_1^2 + q_1^2}{2} + \omega_2 \frac{p_2^2 + q_2^2}{2} = h_1(q_1, p_1) + h_2(q_2, p_2)$$

be the Hamiltonian of a system defined on the phase space $M = \mathbb{R}^4 \setminus \{0 \in \mathbb{R}^4\}$ endowed with the standard symplectic form $\Omega = dp^1 \wedge dq^1 + dp^2 \wedge dq^2$. Here $(\omega_1, \omega_2) \in \mathbb{R}^2$ are the frequencies of two oscillators. Two cases are possible, the resonant and non-resonant one:

- when $\omega_1/\omega_2 \in \mathbb{Q}$ the oscillators are resonant,
- while if $\omega_1/\omega_2 \notin \mathbb{Q}$, they are non-resonant.

In the first case we will see that there are more than 2 first integrals and the system is superintegrable. The functions $\bar{h}_1(q_1, q_2, p_1, p_2) := h_1(q_1, p_1)$ and $\bar{h}_2(q_1, q_2, p_1, p_2) := h_2(q_2, p_2)$, are first integrals in involution. Indeed,

$$\mathcal{L}_{X_H} \bar{h}_1 = \begin{bmatrix} \omega_1 p_1 \\ \omega_2 p_2 \\ -\omega_1 q_1 \\ -\omega_2 q_2 \end{bmatrix} \cdot \begin{bmatrix} \omega_1 q_1 \\ 0 \\ \omega_1 p_1 \\ 0 \end{bmatrix} = 0$$

and similarly for $\bar{h}_2$. Moreover,

$$0 = \{H, \bar{h}_1\} = \{\bar{h}_1 + \bar{h}_2, \bar{h}_1\} = \{\bar{h}_2, \bar{h}_1\},$$

so the two first integrals are in involution. This implies that all the regular invariant level sets

$$\{\boldsymbol{m} \in M : (\bar{h}_1 \times \bar{h}_2)(m) = \boldsymbol{c} \in \mathbb{R}^{>0} \times \mathbb{R}^{>0}\} = \{\bar{h}_1 = c_1\} \times \{\bar{h}_2 = c_2\} \simeq \mathcal{S}^1 \times \mathcal{S}^1$$

are diffeomorphic to $\mathbb{T}^2$. We can consider the two functions τ_1 and τ_2 such that:

$$X_{\bar{h}_i} = \frac{\partial}{\partial \tau_i}, \quad i = 1, 2.$$

With respect to $(\bar{h}_1, \tau_1, \bar{h}_2, \tau_2)$ the vector field X_H reads:

$$X_H = \frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_2}.$$

Indeed, these functions are not well-defined coordinates, since there is not a unique representation of all the points of M. Indeed, for all $\tau_i \in \{t_i + 2\pi/\omega_i\mathbb{Z}\}$ we get the same point on the phase space M. Therefore, we project on the quotient space

$$\pi : (\bar{h}_1, \tau_1, \bar{h}_2, \tau_2) \to (\bar{h}_1, \tau_1 \mod (2\pi/\omega_1), \bar{h}_2, \tau_2 \mod (2\pi/\omega_2)).$$

This projection allows to coordinatize properly the phase space M as

$$\begin{aligned}&\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{S}^1_{2\pi/\omega_1} \times \mathcal{S}^1_{2\pi/\omega_2} \to M, \varphi(a_1, a_2, \alpha_1, \alpha_2) = (q_1, p_1, q_2, p_2)\\ &= \Big(\sqrt{\frac{2a_1}{\omega_1}} \sin(\omega_1\alpha_1), \sqrt{\frac{2a_1}{\omega_1}} \cos(\omega_1\alpha_1), \sqrt{\frac{2a_2}{\omega_2}} \sin(\omega_2\alpha_2), \sqrt{\frac{2a_2}{\omega_2}} \cos(\omega_2\alpha_2)\Big),\end{aligned}$$

which is a global diffeomorphism. This is not a symplectic diffeomorphism just because of a scalar coefficient. To get a symplectomorphism and hence the so called action-angle variables, we just compose the map π with the diffeomorphism

$$\begin{aligned}p &: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{S}^1_{2\pi} \times \mathcal{S}^1_{2\pi} \to \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{S}^1_{2\pi/\omega_1} \times \mathcal{S}^1_{2\pi/\omega_2}\\ p&(a_1, a_2, \alpha_1, \alpha_2) = (a_1/\omega_1, \alpha_1\omega_1, a_2/\omega_2, \alpha_2\omega_2).\end{aligned}$$

Now we can conclude defining the global diffeomorphism $\mathcal{C} := \varphi \circ p$ which reads:

$$\begin{aligned}\mathcal{C}(b_1, b_2, \beta_1, \beta_2) &= (q_1, p_1, q_2, p_2) =\\ &= (\sqrt{2b_1} \sin(\beta_1), \sqrt{2b_1} \cos(\beta_1), \sqrt{2b_2} \sin(\beta_2), \sqrt{2b_2} \cos(\beta_2)).\end{aligned}$$

This is a symplectomorphism with respect to the form $\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ on M and $\bar{\Omega} = db_1 \wedge d\beta_1 + db_2 \wedge d\beta_2$ on the other space. Hence these are action-angle variables for the system and with respect to these coordinates the dynamics is conjugated to the one given by the Hamiltonian vector field of $h := \mathcal{C}^*H = \omega_1 b_1 + \omega_2 b_2$:

$$X_h = \omega_1 \frac{\partial}{\partial\beta_1} + \omega_2 \frac{\partial}{\partial\beta_2}.$$

2.2 Noncommutative integrability: statement and first example

A considerably important number of Hamiltonian systems on symplectic manifolds have more integrals of motion than $n = \frac{1}{2} dim\, M$. This gives rise to a richer description of the dynamics, based on the theory of noncommutative integrability.

Theorem 5 (Noncommutative integrability)**.** *Let (M, Ω) be a $2n-$dimensional symplectic manifold and $H \in \mathcal{C}^\infty(M)$ a smooth Hamiltonian function on M. Assume that there exist $(2n-k) \geq n$ functionally independent first integrals of X_H defining a submersion*

$$F = (F_1, ..., F_{2n-k}) : M \to \mathbb{R}^{2n-k}$$

with compact and connected fibers. Moreover, suppose that there exist functions $P_{ij} : F(M) \to \mathbb{R}$ such that

- $rank(P(F(x))) = 2n - 2k, \quad \forall x \in M,$
- $\{F_i, F_j\} = P_{ij} \circ F, \quad i, j = 1, ..., 2n - k.$

Then any level set N of F is invariant, diffeomorphic to $\mathbb{T}^k$ and it admits a tubular neighbourhood $\mathcal{U}(N)$ which has a locally trivial fibration $\mathcal{U}(N) \simeq \mathbb{T}^k \times B$ structure, with $B \subseteq \mathbb{R}^{2n-k}$. Furthermore, any invariant level set of F restricted to this tubular neighbourhood corresponds to a point in B and, coordinatizing B with

$$(x_1, ..., x_{n-k}, y_1, ..., y_{n-k}, a_1, ..., a_k),$$

the restriction of Ω to $\mathcal{U}(N)$ reads

$$\Omega|_{\mathcal{U}(N)} = \sum_{i=1}^{n-k} dx^i \wedge dy^i + \sum_{j=1}^{k} da^j \wedge d\alpha^j. \tag{2.1}$$

The restriction of the vector field X_H to $\mathcal{U}(N)$ in this coordinate system reads

$$X_H\Big|_{\mathcal{U}(N)} = \sum_{i=1}^{k} \omega_i(a_1, ..., a_k) \frac{\partial}{\partial \alpha_i}.$$

The coordinates which allow to write Ω as in (2.1), are called *generalized action-angle variables*. Let's notice that completely integrable systems are a particular case of noncommutative integrability. Indeed, when $k = n$ and the Poisson structure degenerates to $P_{ij} = 0$ for any i, j, then the hypotheses of Liouville-Arnold Theorem are satisfied. We now analyze again the example of two resonant uncoupled harmonic oscillators, but, this time, in the light of this integrability result.

2.2.1 Superintegrability of two uncoupled resonant oscillators

Consider the Hamiltonian function

$$H(q_1, q_2, p_1, p_2) = \omega_1 \frac{q_1^2 + p_1^2}{2} + \omega_2 \frac{q_2^2 + p_2^2}{2} = h_1(q_1, p_1) + h_2(q_2, p_2),$$

where $\omega_1/\omega_2 \in \mathbb{Q}$ because in this case, due to the previous analysis we have a periodic flow on the torus, so there is a finer geometric structure then the fibration in $2-$tori. Without loss of generality, we assume $\omega_1 = \omega_2 = 1$. Let's verify that the system admits 3 first integrals

$$X_H(F_1) = \mathcal{L}_{X_H}(q_1 q_2 + p_1 p_2) = \begin{bmatrix} q_2 \\ q_1 \\ p_2 \\ p_1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ -q_1 \\ -q_2 \end{bmatrix} = p_1 q_2 + q_1 p_2 - q_1 p_2 - q_2 p_1 = 0,$$

$$X_H(F_2) = \mathcal{L}_{X_H}(p_1 q_2 - p_2 q_1) = \begin{bmatrix} -p_2 \\ p_1 \\ q_2 \\ -q_1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ -q_1 \\ -q_2 \end{bmatrix} = -p_1 p_2 + p_1 p_2 - q_1 q_2 + q_1 q_2 = 0,$$

$$X_H(F_3) = \mathcal{L}_{X_H}\Big(\frac{q_1^2+p_1^2-q_2^2-p_2^2}{2}\Big) = \begin{bmatrix} q_1 \\ -q_2 \\ p_1 \\ -p_2 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ -q_1 \\ -q_2 \end{bmatrix} = 0.$$

These three first integrals define a surjective submersion with compact fibers and we now verify that they satisfy the other assumptions of the noncommutative integrability Theorem. Computing their Poisson brackets we have

$$\begin{aligned}
&\{F_1, F_2\} = \{q_1q_2 + p_1p_2, p_1q_2 - p_2q_1\} = q_2^2 - q_1^2 - (-p_2^2 + p_1^2) = -2F_3,\\
&\{F_1, F_3\} = \frac{1}{2}\{q_1q_2 + p_1p_2, p_1^2 + q_1^2 - q_2^2 - p_2^2\} = q_2p_1 + p_1q_2 - q_1p_2 - q_1p_2 = 2F_2\\
&\{F_2, F_3\} = \frac{1}{2}\{p_1q_2 - p_2q_1, p_1^2 + q_1^2 - p_2^2 - q_2^2\} = -p_2p_1 - p_1p_2 - (q_2q_1 + q_1q_2) = -2F_1,\\
&\{F_1, F_1\} = \{F_2, F_2\} = \{F_3, F_3\} = 0.
\end{aligned}$$

So they can be arranged in the matrix of the Poisson brackets P (the Poisson tensor of the system) which reads:

$$P = \begin{bmatrix} 0 & -2F_3 & 2F_2 \\ 2F_3 & 0 & -2F_1 \\ -2F_2 & 2F_1 & 0 \end{bmatrix} = 2\hat{F},$$

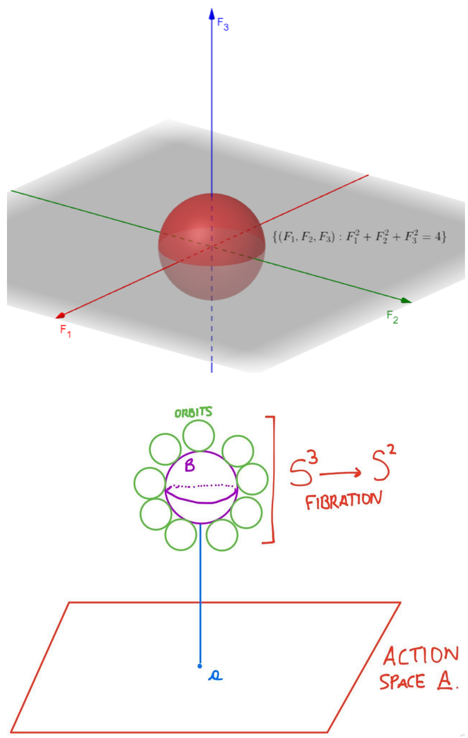
where $F = [F_1, F_2, F_3]^T$. This is the hat matrix of $2F$ so $Pv = 2F \times v$ for any $v \in \mathbb{R}^3$. This means in particular that P is constant on the level sets of F, as required in the noncommutative integrability Theorem. Moreover, since it is a skew-symmetric matrix, then it has necessarily even-rank, so $rank\, P = 2$. The configuration manifold in this case has dimension $n = 2$, the matrix has rank $2 = 4 - 2k$, which implies that we should have $k = 1$. The fact that we have $3 = 2n - k$ independent first integrals hence guarantees that the hypotheses of noncommutative integrability Theorem hold. We can now define two fibrations of the phase space $M = \mathbb{R}^4 \backslash \{0 \in \mathbb{R}^4\}$. The first one is given by the Hamiltonian:

$$H : M \to A,$$

where A is the action space of the system. In this case the action space is $A = \mathbb{R}^{>0}$ and the fibers of H are $3-$dimensional spheres $\mathcal{S}^3$ since we have

$$H(q_1, p_1, q_2, p_2) = \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2) = a \in \mathbb{R}^{>0},$$

so $H^{-1}(a) \simeq \mathcal{S}^3$. On the other hand, we have the fibration given by the three functionally independent first integrals $F = (F_1, F_2, F_3) : M \to B = \mathbb{R}^3$. The regular level sets of F are diffeomorphic to $1-$dimensional tori ($\mathcal{S}^1$) thanks to noncommutative integrability Theorem. Moreover, being connected and of dimension 1, they coincide with the orbits which are hence periodic. The key point now is to project via the fibration F each level set of H onto B. The base space B is coordinatized by $F = (F_1, F_2, F_3)$ and hence in correspondence of a single level set of H, we get a $2-$dimensional sphere. Indeed, on each level set, $\{H = a \in \mathbb{R}^{>0}\}$, the modulus of the vector $F = (F_1, F_2, F_3)$ is constant since $F_1^2 + F_2^2 + F_3^2 = H^2 = a^2$. These $2-$dimensional spheres are the symplectic leaves of the base space B since they coincide with the orbits of the action of $\mathbb{R}^3$ on the



level sets of H generated by the Hamiltonian vector fields $X_{F_1}, X_{F_2}, X_{F_3}$. To be precise, it is not correct in general to say that the level sets of the Hamiltonian admit a product structure, but what happens is that there is a fibration from $\mathcal{S}^3$ to $\mathcal{S}^2$ with fibers coinciding with the orbits of the Hamiltonian vector field X_H, which are diffeomorphic to $\mathcal{S}^1$, i.e. $\mathcal{S}^3 \simeq \mathcal{S}^1 \times \mathcal{S}^2$. This fibration is known with the name of Hopf fibration. Let's highlight that the analysis done in this example follows the approach suggested by Proposition 12.

2.3 Proof of the Theorem

In this section, we prove noncommutative integrability Theorem. Instead of working directly on Theorem 5, thanks to the properties of bifibrations we first of all recover some equivalent results, proving then one of them.

Theorem 6. *Consider a $2n$-dimensional symplectic manifold (M, Ω) where is defined a fibration $F : M \to B$ with compact, connected and isotropic fibers of dimension $k \le n$. Call $\mathcal{F}$ the foliation whose leaves coincide with the fibers of F. Assume the polar $\mathcal{F}^\perp$ of $\mathcal{F}$ does exist. Then*

- *the fibers of F are diffeomorphic to $\mathbb{T}^k$,*
- *any fiber N of F has a tubular neighbourhood $\mathcal{U}(N)$ endowed with generalized action-angle variables.*

We now introduce (and then prove) a proposition which shows why Theorem 6 is equivalent to Theorem 5.

Proposition 13. *Let $G : M \to G(M)$ be a fibration with the same properties of the one in Theorem 5 and call $\mathcal{F}$ the foliation whose leaves coincide with the fibers of G. Then the fibers of G are isotropic and $\mathcal{F}$ admits a polar foliation $\mathcal{F}^{\perp}$. Moreover, any fibration F defined as in Theorem 6 can be described in a neighbourhood of each fiber by $(2n-k)$ functions with the same properties satisfied by the components of G.*

Before proving this proposition, let's highlight how it can help proving Theorem 5. Due to this proposition, we get that the fibration defined by the $2n-k$ functionally independent first integrals of the vector field X_H, is isotropic. Moreover, the foliation $\mathcal{F}$ defined by the level sets of these first integrals admits a polar. Therefore, Theorem 5 implies Theorem 6. Furthermore, every fibration defined as in Theorem 6 can be written in the form of the Theorem 5. Hence, we even get that Theorem 6 implies Theorem 5. The advantage of Theorem 6 is that due to the global structure of its hypotheses we get a better description of the geometrical situation and this is why we have introduced this equivalent result. Let's now prove proposition 13.

Proof. The first integrals of the foliation $\mathcal{F}$ coincide, up to a possible rescaling or combination, with the components of the fibration $G = (G_1, ..., G_{2n-k})$. By assumption we know

$$\{G_i, G_j\} = P_{ij} \circ G,\ i, j = 1, ..., 2n-k.$$

Recalling that this foliation is defined by the submersion G and hence the lift to M via G of any function $g \in \mathcal{C}^{\infty}(G(M))$ is a first integral of $\mathcal{F}$, we conclude the family of first integrals of $\mathcal{F}$ is closed under the Poisson bracket. This is equivalent to the existence of a polar $\mathcal{F}^{\perp}$. Since the polar does exist and the leaves of $\mathcal{F}$ are isotropic submanifolds of M, follows:

$$rank\, P(G) = 2\, dim(G(M)) - dim\, M.$$

Here $P = (P_{ij})$ is the Poisson tensor defined on $G(M)$. This is true in our case since by assumption $rank\, P(G) = 2n - 2k$ and

$$2\, dim(G(M)) - dim\, M = 2(2n-k) - 2n = 2n - 2k.$$

We now move to the second part of the statement.

Consider a point $u \in F(M)$. Define a diffeomorphism $\mathcal{C} : U \to \mathbb{R}^{2n-k}$ where $U \subset B$ is a small-enough open neighbourhood of u. Set

$$G := \mathcal{C} \circ (F|_{F^{-1}(U)}) : F^{-1}(U) \to \mathbb{R}^{2n-k}.$$

Since $\mathcal{C}$ is a diffeomorphism and F is a submersion, up to restricting the arrival space, the map G is a surjective submersion. The level sets of G coincide with the fibers of F by construction. Hence G defines a fibration on $F^{-1}(U)$ (a neighbourhood of a fiber of F). We can define the components of F as the components of G:

$$G = (G_1, ..., G_{2n-k}) = (\mathcal{C}_1 \circ F, ..., \mathcal{C}_{2n-k} \circ F).$$

We conclude verifying these components satisfy the required properties. By assumption, $\mathcal{F}$ admits a polar $\mathcal{F}^{\perp}$. Hence

$$\{G_i, G_j\} = \{\mathcal{C}_i \circ \pi, \mathcal{C}_j \circ \pi\} = \{\mathcal{C}_i, \mathcal{C}_j\} \circ \pi.$$

Furthermore, the fibers of G are isotropic. This implies that the rank of the Poisson tensor $P(G)$ is the correct one.

□

We now conclude the chain of equivalent results with Theorem 7. Indeed, we first prove this Theorem is equivalent to Theorem 6 (and hence even to Theorem 5) and then prove it.

Theorem 7. *Consider a symplectic manifold (M, Ω) of dimension $2n$. Assume there are $k \leq n$ functionally independent functions which are pairwise in involution $F_1, ..., F_k : M \to \mathbb{R}$ (call $F = (F_1, ..., F_k)$). Call $\mathcal{D}$ the tangent distribution generated by the Hamiltonian vector fields $X_{F_1}, ..., X_{F_k}$ and suppose the integral manifolds of $\mathcal{D}$ are compact and the fibers of a fibration π. Then each connected component $N \subset M$ of these integral manifolds is diffeomorphic to $\mathbb{T}^k$ and admits a tubular neighbourhood $\mathcal{U}(N)$ where a set of generalized action-angle variables (a, x, y, α) is well defined. Moreover,*

- *the integral manifolds coincide with the level sets of the map*

$$(a, x, y) : \mathcal{U}(N) \to \mathbb{R}^{2n-k},$$

- *the actions $a_1, ..., a_k$ are functions of $F_1, ..., F_k$.*

This theorem is equivalent to Theorem 6. Precisely, the action of $\mathbb{R}^k$ defined by the Hamiltonian vector fields $X_{F_1}, ..., X_{F_k}$ defines a foliation $\mathcal{F}$ on M. The leaves of $\mathcal{F}$ are isotropic submanifolds since a vector field tangent to these leaves is of the form

$$Y = \sum_{i=1}^{k} g_i X_{F_i}.$$

Indeed, for any $Y, Z \in \mathfrak{X}(N)$, where N is a leaf of $\mathcal{F}$, we have:

$$\Omega(Y, Z) = \sum_{i,j=1}^{k} c_{ij} \Omega(X_{F_i}, X_{F_j}) = \sum_{i,j=1}^{k} c_{ij} \{F_i, F_j\} = 0.$$

Moreover, $\mathcal{F}$ admits a polar foliation and hence Theorem 6 follows from this one. Furthermore, let's assume there is a fibration $\pi : M \to B$ satisfying the hypotheses of Theorem 6. Since the polar foliation $\mathcal{F}$ does exist and the fibers of π are isotropic, follows that the leaves of $\mathcal{F}$ are generated by the Hamiltonian vector fields of the lifts to M of the Casimirs of B. Let $G_1, ..., G_k : B \to \mathbb{R}$ be the Casimirs of B. Since the existence of a polar for the foliation defined by π is equivalent to the fact that π is a Poisson mapping between $(M, \{\cdot, \cdot\}_M)$ and $(B, \{\cdot, \cdot\}_B)$, follows that defining $H_i = G_i \circ \pi$ for any $i = 1, ..., k$:

$$\{H_i, H_j\}_M = \{G_i \circ \pi, G_j \circ \pi\}_M = \{G_i, G_j\}_B \circ \pi = 0.$$

Hence there exist k functionally independent functions $H_1, .., H_k : M \to \mathbb{R}$ in involution. Moreover, the integral manifolds of the distribution generated by the lifted Casimirs coincide exactly with the $k-$dimensional fibers of π, so all the hypotheses of this last theorem hold. Follows that even the second implication is proven and the two theorems in analysis are indeed equivalent. We now prove Theorem 7.

Proof. All the connected components of the fibers of π are $k-$dimensional tori since they all have k independent, tangent and commuting vector fields: $X_{F_1}, ..., X_{F_k}$ (see Proposition 21). Indeed, they commute since $[X_{F_i}, X_{F_j}] = X_{\{F_i,F_j\}} = 0$. Consider a point $m \in M$ and call $N \subset M$ the connected component of the fiber of π where m lives. By Theorem 2 we can complete the set $\{F_1, .., F_k\}$ to a Darboux system of coordinates in a neighbourhood $P \subset M$. In this system of coordinates the symplectic form Ω reads

$$\Omega|_P = \sum_{i=1}^{k} dF_i \wedge dT_i + \sum_{j=1}^{n-k} dX_j \wedge dY_j.$$

We define

- $\mathcal{T} := T(P) \subset \mathbb{R}^k$ and
- $\mathcal{F} := (F \times X \times Y)(P) \subset \mathbb{R}^{2n-k}$,

where $T = T_1 \times ... \times T_k$, $X = X_1 \times ... \times X_{n-k}$ and same for Y. We then coordinatize the whole space P via the inverse map

$$\mathcal{C} := (F \times X \times Y \times T)^{-1} : \mathcal{F} \times \mathcal{T} \to P.$$

In the remaining part of the proof we will denote with (F, X, Y, T) the coordinates on P, while with (f, x, y, t) the associated ones in $\mathcal{F} \times \mathcal{T}$. Precisely:

$$f := \mathcal{C}^* F, \quad x := \mathcal{C}^* X, \quad y := \mathcal{C}^* Y, \quad t := \mathcal{C}^* T.$$

The Hamiltonian vector fields of the F_is are translations along the T_is, namely $X_{F_i} = \partial/\partial T_i$. This implies that each fiber of π with non-trivial intersection with P can be coordinatized by the T_is and it can be written as $(F \times X \times Y)^{-1}(c)$ for some $c \in \mathbb{R}^{2n-k}$. Precisely, $F \times X \times Y : P \to \mathcal{F}$ is a surjective submersion with compact fibers, hence by Ehresmann's Theorem it is a locally-trivial fibration. Namely, if $(F \times X \times Y)(N) = c$ for a certain torus $N \subset M$, then there exists a small-enough open neighbourhood $B \subset \mathcal{F}$ of c such that

$$\mathcal{U}(N) = (F \times X \times Y)^{-1}(B) \simeq N \times U \simeq \mathbb{T}^k \times U,$$

where $\mathcal{U}(N)$ is a tubular neighbourhood of N. With an abuse of notation, we keep calling $\mathcal{F}$ the open set of $\mathbb{R}^{2n-k}$ defining the tubular neighbourhood introduced above. We now define the section

$$\sigma : \mathcal{F} \to P, \, \sigma(b) := \mathcal{C}(b, 0)$$

and assume P to be small enough so that each invariant torus, with non-trivial intersection with P, intersects $\sigma(\mathcal{F})$ in exactly one point (this can be done due to the locally-trivial fibration introduced above). $\sigma(\mathcal{F}) \subset M$ is an embedded submanifold of M of dimension $2n-k$. The X_{F_i}s define an action of $\mathbb{R}^k$ on each

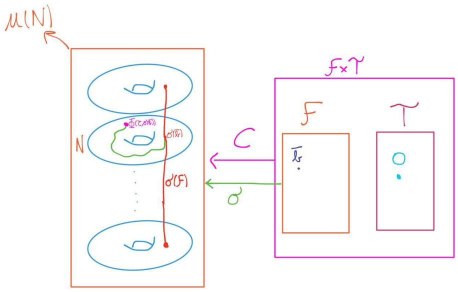


Figure 2.1: Visual description of the involved maps

torus N. We call it Φ. Thanks to this action, we can coordinatize the tubular neighbourhood $\mathcal{U}(N)$ of N as follows:

$$\mathcal{C}(b, \tau) := \Phi(\tau, \sigma(b)).$$

Assume $\sigma(\bar{b}) = \mathcal{C}(\bar{b}, 0) = m$.

Proposition 14. *The map $\mathcal{C} : \mathcal{F} \times \mathcal{T} \to M$ is a local diffeomorphism and it is symplectic with respect to the standard symplectic form*

$$\Omega_s = \sum_{i=1}^{k} df_i \wedge dt_i + \sum_{j=1}^{n-k} dx_j \wedge dy_j.$$

Proof. To show that it is a local diffeomorphism, we verify that the columns of the Jacobian matrix $J\mathcal{C}$ of $\mathcal{C}$ are linearly independent in a neighbourhood of m. We then conclude by inverse function theorem. We have already seen that

$$\frac{\partial \mathcal{C}}{\partial t_i}(\bar{b}, 0) = \frac{\partial \Phi}{\partial t_i}(0, \sigma(\bar{b})) = X_{F_i}(\Phi(0, \sigma(\bar{b})) = X_{F_i}(\mathcal{C}(\bar{b}, 0)) = X_{F_i}(m).$$

Taking then the derivative with respect to the other variables and evaluating them at $(\bar{b}, 0)$ we get

$$\frac{\partial \sigma}{\partial f_1}(\bar{b}), ..., \frac{\partial \sigma}{\partial f_k}(\bar{b}), \frac{\partial \sigma}{\partial x_1}(\bar{b}), ..., \frac{\partial \sigma}{\partial x_{n-k}}(\bar{b}), \frac{\partial \sigma}{\partial y_1}(\bar{b}), ..., \frac{\partial \sigma}{\partial y_{n-k}}(\bar{b}).$$

Since σ is transversal to $N = (F \times X \times Y)^{-1}(\bar{b})$, we conclude that they are all linearly independent vectors and hence $\mathcal{C}$ is a local diffeomorphism. We now need to check that the map $\mathcal{C} : \mathcal{F} \times \mathcal{T} \to P$ is symplectic. This amounts to prove that $\mathcal{C}^*(\Omega|_P) = \Omega_s$.

$$\Omega|_P = \sum_{i=1}^{k} dF_i \wedge dT_i + \sum_{j=1}^{n-k} dX_j \wedge dY_j$$

implies:

$$\begin{aligned}
\mathcal{C}^*\Omega|_P &= \sum_{i=1}^{k} d(\mathcal{C}^*F_i)\wedge d(\mathcal{C}^*T_i) + \sum_{j=1}^{n-k} d(\mathcal{C}^*X_j)\wedge d(\mathcal{C}^*Y_j) = \\
&= \sum_{i=1}^{k} d(F_i\circ\mathcal{C})\wedge d(T_i\circ\mathcal{C}) + \sum_{j=1}^{n-k} d(X_j\circ\mathcal{C})\wedge d(Y_j\circ\mathcal{C}) = \\
&= \sum_{i=1}^{k} df_i\wedge dt_i + \sum_{j=1}^{n-k} dx_j\wedge dy_j = \Omega_s.
\end{aligned}$$

□

We can restrict $\mathcal{C}$ to $\mathcal{A}\times\mathcal{B}\subset\mathcal{F}\times\mathcal{T}$ in such a way that

$$\mathcal{C}:\mathcal{A}\times\mathcal{B}\to\mathcal{C}(\mathcal{A}\times\mathcal{B})=\mathcal{U}(N)$$

is a diffeomorphism. What remains to do at this point is to transform (f,x,y,t) into $(a,x,y,\hat{\alpha})$ via a symplectomorphism on $\mathcal{A}\times\mathbb{R}^k$ and hence define the generalized action-angle variables. On each invariant torus

$$(F\times X\times Y)^{-1}(b),$$

all the points have the same period lattice and matrix with respect to the action given by the X_{F_i}s (because of what we will prove in Proposition 21). We call them respectively G_b and $L(b)$.

Proposition 15. *Consider a pair of points $m=\mathcal{C}(b_1,\tau_1)$ and $n=\mathcal{C}(b_2,\tau_2)$ in $\mathcal{U}(N)$. Then $G_{b_1}=G_{b_2}$ when $F(m)=F(n)$.*

Proof. In this proof we need two group actions. One consists in $(\mathbb{R}^{2n-k},+)$ acting on M and is defined as

$$\theta:\mathbb{R}^{2n-k}\times\mathcal{U}(N)\to\mathcal{U}(N),$$

$$\theta_s:=\Phi_{s_1}^{X_{F_1}}\circ\ldots\circ\Phi_{s_k}^{X_{F_k}}\circ\Phi_{s_{k+1}}^{X_{X_1}}\circ\ldots\circ\Phi_{s_n}^{X_{X_{n-k}}}\circ\Phi_{s_{n+1}}^{X_{Y_1}}\circ\ldots\circ\Phi_{s_{2n-k}}^{X_{Y_{n-k}}}.$$

The second action is the usual one of $\mathbb{R}^k$ over $\mathcal{U}(N)$:

$$\Phi_\tau:=\Phi_{\tau_1}^{X_{F_1}}\circ\ldots\circ\Phi_{\tau_n}^{X_{F_k}}.$$

$X_{F_i}=\partial/\partial T_i$ commutes with all the other Hamiltonian vector fields of the type X_{F_j}, X_{X_h} and X_{Y_l}. This implies that:

$$\Phi_\tau\circ\theta_s=\theta_s\circ\Phi_\tau,\quad\forall\tau\in\mathbb{R}^k,\ \forall s\in\mathbb{R}^{2n-k}.$$

Let's recall that $\bar{\tau}\in G_{b_1}$ if and only if $\Phi_{\bar{\tau}}(m)=m$. This implies that

$$\Phi_{\bar{\tau}}(\theta_t(m))=\theta_s(\Phi_{\bar{\tau}}(m))=\theta_s(m).$$

Hence if $n=\theta_s(m)$, then $G_{b_1}=G_{b_2}$. Now what remains to verify is that $\theta_s(m)=n$ for some $s\in\mathbb{R}^{2n-k}$ if and only if $F(m)=F(n)$.

$$X_{F_j}=\frac{\partial}{\partial T_j},\quad X_{X_h}=\frac{\partial}{\partial Y_h},\quad X_{Y_l}=-\frac{\partial}{\partial X_l},$$

so the action θ leaves unchanged the first k components, those associated to the coordinates $F_1, ..., F_k$. This means that it is possible to have $\theta_t(m) = n$ if and only if $F(m) = F(n)$, since on these level sets the action of θ is transitive.

$\square$

We can hence denote the period lattice and matrix with G_f and $L(f)$ respectively. Recall that the period matrix has the following entries

$$L_{ij}(f) = u_j(f) \cdot e_i,$$

where $\{u_j(f)\}_{j=1}^k$ is the basis of the period lattice. Now we uniform the period lattices of all the fibers $F^{-1}(f)$ intersecting $\mathcal{U}(N)$. To do so, we introduce the angular coordinate $\hat{\alpha} = L^{-1}(f)\, t$, similarly to what we will do to prove Bogoyavlensky's Theorem. Moreover, to complete the symplectic diffeomorphism ψ defining the generalized action-angle variables, remains to find an action function $a = a(f)$ in such a way that

$$\Omega_s = \sum_{i=1}^{k} df_i \wedge dt_i = \psi^* \Omega = \sum_{i=1}^{k} da_i(f) \wedge d\hat{\alpha}_i.$$

To get an explicit condition which guarantees the validity of this expression, we compute

$$da_i \wedge d\hat{\alpha}_i = \sum_{j=1}^{k} \frac{\partial a_i(f)}{\partial f_j} df_j \wedge d\hat{\alpha}_i.$$

Moreover, $t_i = \sum_{j=1}^{k} L_{ij}(f)\hat{\alpha}_j$, so

$$\begin{aligned} df_i \wedge d\tau_i &= df_i \wedge d\Big(\sum_{j=1}^{k} L_{ij}(f)\hat{\alpha}_j\Big) \\ &= df_i \wedge \Big[\Big(\sum_{j,h=1}^{k} \frac{\partial L_{ij}(f)}{\partial f_h}\hat{\alpha}_j df_h\Big) + \Big(\sum_{j=1}^{k} L_{ij}(f) d\hat{\alpha}_j\Big)\Big] = \\ &= \Big[\Big(\sum_{j,h=1}^{k} \frac{\partial L_{ij}(f)}{\partial f_h}\hat{\alpha}_j df_i \wedge df_h\Big) + \Big(\sum_{j=1}^{k} L_{ij}(f) df_i \wedge d\hat{\alpha}_j\Big)\Big]. \end{aligned}$$

This implies necessarily

$$\frac{\partial L_{ij}(f)}{\partial f_h}\hat{\alpha}_j - \frac{\partial L_{hj}(f)}{\partial f_i}\hat{\alpha}_j = 0 \quad \forall i, j, h = 1, ..., k, \tag{2.2}$$

$$L_{ji}(f) = \frac{\partial a_i(f)}{\partial f_j} \quad \forall i, j = 1, ..., k. \tag{2.3}$$

The condition (2.2) can be seen as the closure condition of the $1-$form

$$\beta_j = \sum_{i=1}^{k} L_{ij}(f) df_i,$$

and when it holds, (2.3) can be integrated locally and hence we can find the action functions $a_i = a_i(f)$ up to restricting the domain $\mathcal{A}$.

Proposition 16. *The period matrix satisfies the symmetry property in Equation (2.2) if and only if the section $\sigma : \mathcal{A} \to P$ is co-isotropic.*

Proof. The action Φ generated by the Hamiltonian vector fields $X_{F_1}, ..., X_{F_k}$ is not free and, supposing $m \in F^{-1}(f)$, for any $\nu \in \mathbb{Z}^k$ we have $m = \Phi(L(f)\nu, m)$. Hence the inverse images under $\mathcal{C}$ of the submanifold $\sigma(\mathcal{A})$ are defined by the submanifolds of $\mathcal{A} \times \mathbb{R}^k$ given by

$$\Sigma_\nu : (f, x, y) \to (f, x, y, L(f)\nu), \quad \nu \in \mathbb{Z}^k,$$

since all the times $L(f)\nu$ map $\sigma(f, x, y)$ to the same point of the Liouville torus $(F \times X \times Y)^{-1}(f, x, y)$. This construction is helpful since we have previously proven that $\mathcal{C}$ is symplectic and hence it sends co-isotropic submanifolds into co-isotropic submanifolds. Therefore, we check what has to happen to have that, for at least a $\nu \in \mathbb{Z}^k$, $\Sigma_\nu(\mathcal{A})$ is co-isotropic. This is equivalent to prove that for any $(f, x, y, t) \in \Sigma_\nu(\mathcal{A})$, $(T_{(f,x,y,t)}\Sigma_\nu(\mathcal{A}))^\Omega$ is isotropic for this particular $\nu \in \mathbb{Z}^k$. The dimension of $\Sigma_\nu(\mathcal{A})$ is $(2n - k)$. We consider

$$X_j = \frac{\partial}{\partial x_j},\ Y_j = \frac{\partial}{\partial y_j},\ W_i = \frac{\partial}{\partial f_i} + \sum_{r,s=1}^{k} \frac{\partial L_{rs}}{\partial f_i}\nu_s \frac{\partial}{\partial t_r},\ i = 1, ..., k,\ j = 1, ..., n-k$$

where

$$\sum_{r,s=1}^{k} \frac{\partial L_{rs}}{\partial f_i}\nu_s \frac{\partial}{\partial t_r} = \sum_{r=1}^{k} \frac{\partial}{\partial f_i}\Big(\sum_{s=1}^{k} L_{rs}\nu_s\Big)\frac{\partial}{\partial t_r}$$

is a linear combination of independent vector fields with coefficients given by the $i-$th column of the Jacobian matrix of $L(f)\nu$. Since L is an isomorphism, all the W_is are independent as desired. Hence a vector V belongs to the symplectic complement of this space if and only if it is of the form

$$V = \sum_{i=1}^{k} \lambda_i W_i,$$

noticing that necessarily the components along the X_js and Y_js vanish. This means that $(T_{(f,x,y,t)}\Sigma_\nu(\mathcal{A}))^\Omega$ is isotropic if and only if $\Omega(W_i, W_j) = 0$ for any $i, j = 1, .., k$.

- $df_h(W_i) = \delta_{ih}$,
- $dt_h(W_i) = \sum_{s=1}^{k} \frac{\partial L_{hs}}{\partial f_i}\nu_s$.

This implies that:

$$(df_h \wedge dt_h)(W_i, W_j) = \delta_{ih}\sum_{s=1}^{k}\frac{\partial L_{hs}}{\partial f_i}\nu_s - \delta_{jh}\sum_{s=1}^{k}\frac{\partial L_{hs}}{\partial f_j}\nu_s$$

Hence we conclude

$$\Omega(W_i, W_j) = \sum_{h=1}^{k}(df_h \wedge dt_h)(W_i, W_j) = \sum_{s=1}^{k}\Big(\frac{\partial L_{is}}{\partial f_i} - \frac{\partial L_{js}}{\partial f_j}\Big)\nu_s = 0$$

if and only if the condition (2.2) holds (indeed, take ν with all the components different from zero). □

The section $\sigma : \mathcal{A} \to \mathcal{U}(N)$ is co-isotropic since the tangent space to $\sigma(\mathcal{A})$ is generated by the vector fields

$$\frac{\partial}{\partial F_1}, \ldots, \frac{\partial}{\partial F_k}, \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_{n-k}}, \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_{n-k}}.$$

Indeed, the tangent space of the symplectic complement of $\sigma(\mathcal{A})$ is generated by

$$\frac{\partial}{\partial F_1}, \ldots, \frac{\partial}{\partial F_k},$$

and Ω vanishes on this space. We therefore get that $\sigma(\mathcal{A})$ is co-isotropic as desired. So we are now ready to conclude the proof, since we have just built a symplectic diffeomorphism

$$\psi : \mathcal{A} \times \mathbb{R}^k \to \mathcal{U} \times \mathbb{R}^k, \ (f, x, y, t) \to (a, x, y, \alpha)$$

up to restricting $\mathcal{A}$ to a small enough open subset. Projecting with the diffeomorphism

$$p : \mathcal{U} \times \mathbb{R}^k \to \mathcal{U} \times \mathbb{T}^k, \ (a, x, y, \alpha) \to (a, x, y, \alpha \mod 1)$$

we get the desired diffeomorphism between $\mathcal{U}(N)$ and $\mathcal{U} \times \mathbb{T}^k$, i.e. the locally trivial fibration of the tubular neighbourhood of N:

$$\mathcal{U}(N) \xrightarrow[\simeq]{F \times X \times Y \times T} \mathcal{A} \times \mathcal{B} \xrightarrow[\simeq]{\psi} \mathcal{U} \times \mathbb{R}^k \xrightarrow[\simeq]{p} \mathcal{U} \times \mathbb{T}^k,$$

where $\mathcal{U}(N) = \mathcal{C}(\mathcal{A} \times \mathcal{B})$.

□

Chapter 3

Bogoyavlensky's integrability Theorem

In this Chapter, we present Bogoyavlensky's integrability Theorem for non-Hamiltonian dynamical systems. It gives the standard definition of integrability of a vector field $X \in \mathfrak{X}(M)$ defined on a smooth $n-$dimensional manifold M. This Theorem relies on the existence of n tensor invariants which are either symmetry fields or integrals of motion. For this reason, before moving to the statement and proof of the Theorem, we now properly define the notion of dynamical symmetry (i.e. symmetry field) and of (Lie) group action.

3.1 Dynamical symmetries and group actions

Let M be a smooth $n-$dimensional manifold. We can define the Lie-algebra of vector fields $(\mathfrak{X}(M), [\cdot, \cdot])$ where

$$[X, Y] = XY - YX$$

is the (Jacobi-)Lie bracket, namely it is a map $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ with the following properties:

1. it is bilinear and skew-symmetric,
2. it satisfies Jacobi's identity, i.e.

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

We say that two vector fields $X, Y \in \mathfrak{X}(M)$ *commute* if their Jacobi-Lie bracket vanishes, i.e. $[X, Y] = 0$. The Jacobi-Lie bracket has even a dynamical interpretation since

$$[X, Y] = \mathcal{L}_X Y.$$

For this reason, we say that if X and Y commute then Y is a *dynamical symmetry* for X (and vice versa).

Proposition 17 (Characterization of commuting fields). *Let $X, Y \in \mathfrak{X}(M)$, where M is a $n-$dimensional smooth manifold. They commute if and only if*

$$\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X \quad \forall t, s$$

which make the computation reasonable, where $\Phi_t^X, \Phi_s^Y : \mathbb{R} \times M \to M$ are the flows of the two fields. This is equivalent to have

$$(\Phi_s^Y)_* X = X \quad \forall s.$$

Proof. We start assuming $(\Phi_s^Y)_* X = X$. By definition,

$$0 = \frac{d}{ds} X = \frac{d}{ds}(\Phi_s^Y)_* X = (\Phi_s^Y)_*[X, Y].$$

This implies $[X, Y] = 0$ as expected since the push-forward is a linear map. Vice versa, if we suppose $[X, Y] = 0$, then

$$\frac{d}{ds}(\Phi_s^Y)_* X = 0,$$

i.e. $(\Phi_s^Y)_* X = (\Phi_0^Y)_* X = X$. □

The flow of a vector field can be seen as the action of the additive group $(\mathbb{R}, +)$ on the manifold M, which is a particular case of Lie group action. We recall the pair $(G, \cdot)$ is a Lie group if it is an algebraic group and G is a smooth manifold, where $\cdot$ and the inversion map are compatible with the differentiable structure of G, i.e.

- $\cdot : G \times G \to G, \quad (a, b) \to a \cdot b,$
- $a \to (a)^{-1}$

are differentiable.

Definition 14 (Lie group action)**.** Let M be a smooth manifold and $(G, \cdot)$ be a Lie group. The (left) action of the group G on M is a map

$$\varphi : G \times M \to M, \quad \varphi(g, m) = \varphi_g(m)$$

such that:

1. $\varphi(1_G, m) = m \quad \forall m \in M,$
2. for any $g \in G$ the map $\varphi_g : M \to M$ is a diffeomorphism and
3. $\forall g, h \in G,\ \varphi_g \circ \varphi_h(m) = \varphi_{g \cdot h}(m).$

Let φ be the action of $(G, \cdot)$ on M.

- The *orbit* of $m \in M$ is $Orb(m) = \{\varphi_g(m) : g \in G\} \subseteq M$.
- The *isotropy subgroup* of $m \in M$ is $G_m = \{g \in G : \varphi_g(m) = m\}$.
- φ is a free action if all the isotropy groups are trivial, i.e. $\varphi_g(m) = m$ if and only if $g = 1_G$.
- It is a transitive action if φ_g is surjective for any $g \in M$, i.e. for any pair $m, n \in M$, there exists $g \in G$ such that $\varphi_g(m) = n$.
- It is a proper action if the map $\psi : G \times M \to M \times M$, $\psi(g, m) = (\varphi_g(m), m)$ is proper.

A Lie group G is said to be a symmetry-group for the vector field $X \in \mathfrak{X}(M)$ if the action $\varphi : G \times M \to M$ commutes with the flow of X. Namely, if $\Phi : \mathbb{R} \times M \to M$ is the flow of X, then $\varphi_g \circ \Phi_t = \Phi_t \circ \varphi_g$ for any $t \in \mathbb{R}$, $g \in G$.

A group action $\psi : G \times M \to M$ can be seen even as the flow of a single vector field, called *infinitesimal generator* of the action. To define this vector field we need to introduce some preliminary definition. A one-parameter subgroup of G is a differentiable map $\sigma : \mathbb{R} \to G$ which is even a homomorphism between the groups $(\mathbb{R}, +)$ and $(G, \cdot)$. The image of a one-parameter subgroup of G is a differentiable curve in G. We can define the notion of Lie-algebra of the group G and we will denote it with $\mathfrak{g}$. This Lie-algebra is defined by the tangent space at the identity T_eG endowed with the commutator

$$[\xi, \eta] := [X_\xi, X_\eta](e) \quad \forall \xi, \eta \in \mathfrak{g}.$$

Here on the right hand side we consider the Jacobi-Lie bracket and X_ξ is called the *left extension* of $\xi \in \mathfrak{g}$ and is defined as

$$X_\xi(g) := T_e\psi_g(\xi).$$

The one-parameter subgroup of G associated to $\xi \in \mathfrak{g}$ can be now defined as the integral curve $\sigma_\xi : \mathbb{R} \to G$ of X_ξ with $\sigma_\xi(0) = e$, $\sigma'_\xi(0) = \xi$. The basic map which allows to associate $\mathfrak{g}$ with G is the exponential map:

$$exp_G : \mathfrak{g} \to G, \ exp_G(\xi) := \sigma_\xi(1).$$

This map allows to describe all the one-parameter subgroups in a simpler way as

$$\{exp(t\xi) : \ t \in \mathbb{R}\} \subset G,$$

and hence the exponential map sends lines of $\mathfrak{g}$ into differentiable curves of G. We now define the infinitesimal generator of the Lie-group action

$$\psi : G \times M \to M, \ (g, m) \to \psi_g(m) := g \cdot m$$

associated to the element $\xi \in \mathfrak{g}$ as the following vector field on M:

$$\xi_M(m) = \frac{d}{dt}(exp_G(t\xi) \cdot m)\Big|_{t=0} \in T_mM.$$

Consider now $X \in \mathfrak{X}(M)$ with flow $\Phi : \mathbb{R} \times M \to M$. The infinitesimal generator of this group action coincides with the vector field X itself. Precisely, in this case $exp_{\mathbb{R}}(t\xi) = t\xi$ for any $t \in \mathbb{R} = G, \xi \in \mathfrak{g} \simeq \mathbb{R}$ and if we set $\xi = 1$

$$\xi_M(m) = \frac{d}{dt}\Phi(\exp_{\mathbb{R}}(\xi t), m)\Big|_{t=0} = \frac{d}{dt}\Phi(\xi t, m)\Big|_{t=0} = X(\Phi(t, m))|_{t=0} = X(m).$$

Furthermore, any action of $(\mathbb{R}, +)$ can be seen as the flow of a vector field, which is precisely its infinitesimal generator.

The reason why symmetry groups are important to analyze dynamical systems, is because they allow to study a reduced version of the system, instead of the original one. Indeed, consider a vector field $X \in \mathfrak{X}(M)$ which is invariant under the group action $\varphi : G \times M \to M$. We can define the projection map $\pi : M \to M/G$ introducing the following equivalence relation on M:

$$m_1, m_2 \in M, m_1 \sim m_2 \iff \varphi(g, m_2) = m_1$$

for some $g \in G$. M/G is called the orbit space and we endow it with the quotient topology.

Proposition 18. *Assume $\varphi : G \times M \to M$ is a free and proper Lie-group action with $n = dim\, M \geq k = dim\, G$. Then M/G is a smooth manifold (thanks to Quotient Manifold's Theorem whose proof can be found in [6]). Furthermore, the orbits of φ are $k-$dimensional submanifolds of M diffeomorphic to G and $\pi : M \to M/G$ is a submersion, where $dim\,(M/G) = n - k$.*

In general the push-forward of a vector field is not a vector field, but in this setting with φ free and proper everything works fine. Indeed, we can define $\tilde{X} = \pi_* X \in \Gamma(T(G/M))$ the vector field on M/G such that

$$\Phi_t^{\tilde{X}} \circ \pi = \pi \circ \Phi_t^X, \quad \forall t \in \mathbb{R}.$$

The pair $(M/G, \tilde{X})$ is the reduced system and describes the dynamics of X transversally to the orbits of φ. When we are able to compute the integral curves of the reduced vector field $\tilde{X}$, we can then study the dynamics along the orbits of φ via a reconstruction procedure. The reconstruction part consists in computing the integral of a known function if we are able to explicitly find the integral curves of the reduced vector field $t \to r_t$. Indeed, this second part is important to describe the dynamics of X along the orbits of φ.

We will use mainly group actions of $(\mathbb{R}, +)$ or of $(\mathcal{S}^1, +)$ on M. Such actions can be seen as flows of their infinitesimal generators. For this reason, we will always refer to flows of vector fields instead of group actions.

Proposition 19. *Let M be a smooth manifold of dimension n and $Y_1, ..., Y_k$ be tangent, independent and commuting vector fields on M ($1 \leq k \leq n$), namely $[Y_i, Y_j] = 0\ \forall i, j = 1, ..., k$. Then the map*

$$\Phi : \mathbb{R}^k \times M \to M, \quad \Phi(\boldsymbol{\tau}, m) = \Phi_{\tau_1}^{Y_1} \circ ... \circ \Phi_{\tau_k}^{Y_k}(m)$$

is a group action of $(\mathbb{R}^k, +)$ on M.

Proof. First of all $\Phi(\mathbf{0}, m) = m$. Then

$$\Phi(\boldsymbol{\tau} + \boldsymbol{\mu}, m) = \Phi_{\tau_1+\mu_1}^{Y_1} \circ ... \circ \Phi_{\tau_k+\mu_k}^{Y_k}(m) = \Phi_{\tau_1}^{Y_1} \circ ... \circ \Phi_{\tau_k}^{Y_k} \circ \Phi_{\mu_1}^{Y_1} \circ ... \circ \Phi_{\mu_k}^{Y_k}(m)$$

since the vector fields commute and hence even their flows do. By definition, this amounts to $\Phi_{\boldsymbol{\tau}} \circ \Phi_{\boldsymbol{\mu}}(m)$ as desired. The fact that the map $\Phi_{\boldsymbol{\tau}} : M \to M$ is a diffeomorphism, comes because it is the composition of diffeomorphisms. ☐

Proposition 20 (Action of $\mathcal{S}^1$)**.** *Consider a smooth $n-$dimensional manifold M and a vector field $Y \in \mathfrak{X}(M)$ with flow $\Phi_Y : \mathbb{R} \times M \to M$. If Y has all periodic orbits with the same period $T > 0$, then it generates an action of $(\mathcal{S}_T^1, +)$ over M defined as*

$$\varphi : \mathcal{S}_T^1 \times M \to M,\ \varphi(t \mod T, m) := \Phi_Y(t, m)\ \forall t \in \mathbb{R}.$$

With $\mathcal{S}^1_T$ we refer to $\mathbb{R}/(T\mathbb{Z})$.

Proof. The map φ is well defined by the periodicity of the flow of Y, namely

$$\Phi(t+kT,m)=\Phi(t,m)=\Phi(t+kT \mod T,m) \text{ for all } k\in\mathbb{Z}.$$

$\varphi(0,m)=\Phi_Y(0,m)=m$ since Φ_Y is a group action. Then for any pair $s,t\in\mathbb{R}$ follows:

$$\begin{aligned}&\varphi((t+s) \mod T,m)=\Phi_Y(t+s,m)=\Phi_Y(t,\Phi_Y(s,m))=\\ &=\Phi_Y(t,\varphi(s \mod T,m))=\varphi(t \mod T,\varphi(s \mod T,m)).\end{aligned}$$

Φ_Y is a diffeomorphism, so even φ has this property. This is enough to prove that φ is an action of $\mathcal{S}^1_T$ on M . □

3.2 Statement of Bogoyavlensky's Theorem

The standard notion of integrability for non-Hamiltonian vector fields is rather recent and it is due to Bogoyavlensky [17]. We now state this Theorem and then define a class of dynamical systems to be integrable thanks to this result.

Theorem 8 (Broad integrability)**.** *Let M be a smooth manifold of dimension n and $X\in\mathfrak{X}(M)$ a vector field. Let $0<k\leq n$ and*

$$F=(f_1,...,f_{n-k}):M\to\mathbb{R}^{n-k}$$

be a submersion whose components are functionally independent integrals of motion of X. Suppose its level sets $N=\{m\in M:\ F(m)=\boldsymbol{f}\}$ are compact and connected. Moreover, let $Y_1,...,Y_k\in\mathfrak{X}(M)$ be linearly independent dynamical symmetries of X, which pairwise commute and preserve the first integrals of X, i.e.

- $\mathcal{L}_{Y_i}X=[Y_i,X]=0,\ i=1,...,k,$
- $[Y_i,Y_j]=0,\ i,j=1,...,k,$
- $\mathcal{L}_{Y_i}f_j=0$, *for all* $i=1,...,k,\ j=1,...,n-k,$
- $Y_1\wedge...\wedge Y_k\neq 0$ *on* N.

Then N is diffeomorphic to the $k-$dimensional torus $\mathbb{T}^k$ and X is conjugated to a linear flow on it, namely we can write it as

$$\begin{aligned}\dot{f}_i&=0, \quad i=1,...,n-k\\ \dot{\alpha}_j&=\omega_j, \quad j=1,...,k,\ \omega_j\in\mathbb{R}\end{aligned}$$

where $(\alpha_1,...,\alpha_k)$ are angular coordinates on N. Moreover, each invariant torus N admits a neighbourhood $\mathcal{U}(N)$ and a diffeomorphism

$$\varphi:\mathcal{U}(N)\to F(\mathcal{U}(N))\times\mathbb{T}^k,\ \ \varphi(m)=(f(m),\alpha(m))$$

conjugating topologically the vector field X to the following system

$$\begin{aligned}\dot{f}_j&=0, \quad j=1,...,n-k\\ \dot{\alpha}_i&=\omega_i(\boldsymbol{f}), \quad i=1,...,k\end{aligned}$$

on $\mathcal{U}(N)$ where the $\omega_i:F(\mathcal{U}(N))\to\mathbb{R}$ are differentiable functions.

As a consequence of this Theorem, we call the $n-$tuple $(Y_1, ..., Y_p, f_1, ..., f_q)$ an integrable dynamical system of type (p, q), or a $(p, q)-$integrable system, defined over the $n-$dimensional manifold M, if

- $Y_i \in \mathfrak{X}(M), \quad \forall i = 1, ..., p,$
- $Y_1 \wedge ... \wedge Y_p \neq 0$ and $df_1 \wedge ... \wedge df_q \neq 0$,
- $F = (f_1, ..., f_q) : M \to \mathbb{R}^q$ is a submersion,
- $p + q = n$, $0 < p \leq n$, $0 \leq q < n$,
- $[Y_i, Y_j] = 0, \quad \forall i, j = 1, ..., p,$
- $\mathcal{L}_{Y_j} f_k = 0, \quad \forall j = 1, ..., p, \ \forall k = 1, ..., q.$

Hence, a given vector field $X \in \mathfrak{X}(M)$ is integrable if it belongs to some $n-$tuple as above. We will often refer to dynamical systems integrable in this sense as *B-integrable* or *broadly-integrable* systems.

3.3 Example of B-integrable system

We report here an example of integrable system of type $(2, 2)$. Consider the vector field

$$X = p_x\partial_x + p_y\partial_y - \frac{yp_xp_y}{1 + y^2}\partial_{p_x} - y\partial_{p_y}$$

defined on the phase space $M = \mathcal{S}^1 \times \mathbb{R} \times \mathbb{R}^2$. This system admits 2 independent symmetry fields which are

$$X, \quad Y = \partial_x.$$

The coordinate x is cyclic and hence $[X, Y] = 0$. Moreover, X has two integrals of motion which are

- $E = \frac{1}{2}[p_x^2(1 + y^2) + p_y^2] + \frac{1}{2}y^2,$
- $J = p_x\sqrt{1 + y^2}.$

Let's verify that $\mathcal{L}_X E = \mathcal{L}_X J = 0$ here:

$$\mathcal{L}_X E = y(p_x^2 + 1)p_y - p_x(1 + y^2)\frac{yp_xp_y}{1 + y^2} - yp_y = 0,$$
$$\mathcal{L}_X J = \frac{p_xy}{\sqrt{1 + y^2}}p_y - \sqrt{1 + y^2}\frac{yp_xp_y}{1 + y^2} = 0.$$

These two first integrals of X are preserved by Y since both E and J do not depend on x. This means that the hypotheses of Bogoyavlensky's Theorem hold and hence this vector field is $(2, 2)-$integrable.

3.4 Proof of Bogoyavlensky's Theorem

In this section we prove the Theorem 8.

Proof. The map F is a submersion and, moreover, by Ehresmann's Theorem it defines a locally-trivial fibration. This implies that for any regular value $\boldsymbol{c} \in \mathbb{R}^{n-k}$, $F^{-1}(\boldsymbol{c})$ is a $k-$dimensional invariant embedded submanifold $N \subset M$. By assumption, these fibers are compact and connected. Since the k independent vector fields $Y_1, ..., Y_k$ preserve the first integrals of X, i.e.

$$\mathcal{L}_{Y_i} f_j = 0, \quad \forall i = 1, ..., k, \, \forall j = 1, .., n-k,$$

then they are all tangent to the fibers of F.

Proposition 21. *Consider a smooth, compact and connected $n-$dimensional manifold M. Suppose $Y_1, ..., Y_n \in \mathfrak{X}(M)$ are n linearly independent and pairwise commuting tangent vector fields. Then M is diffeomorphic to $\mathbb{T}^n$.*

Proof. We have

$$[Y_i, Y_j] = 0, \quad \forall i, j = 1, ..., n.$$

By Proposition 19, they define the following group action of $(\mathbb{R}^n, +)$:

$$\Phi : \mathbb{R}^n \times M \to M, \; \Phi(\boldsymbol{t}, m) := \Phi_{t_1}^{Y_1} \circ ... \circ \Phi_{t_n}^{Y_n}(m),$$

with $\boldsymbol{t} = (t_1, ..., t_n) \in \mathbb{R}^n$. We can now define the following map

$$\psi_m : \mathbb{R}^n \to M, \; \psi_m(\boldsymbol{t}) := \Phi(\boldsymbol{t}, m), \quad \forall m \in M.$$

Lemma 1. *For any $m \in M$, the map ψ_m is a surjective local diffeomorphism.*

Proof. To prove it is a local diffeomorphism, we write the Jacobian matrix of ψ_m and check that it has full rank.

$$\begin{aligned}\frac{\partial \psi_m}{\partial \tau_1}(\tau) &= \frac{\partial \Phi_{\tau_1}^{Y_1}}{\partial \tau_1} \circ \Phi_{\tau_2}^{Y_2} \circ ... \circ \Phi_{\tau_n}^{Y_n}(m) = Y_1(\Phi_{\tau_1}^{Y_1} \circ \Phi_{\tau_2}^{Y_2} \circ ... \circ \Phi_{\tau_n}^{Y_n}(m)) = \\ &= Y_1(\Phi(m, \tau)).\end{aligned}$$

Since all the fields pairwise commute, so do their flows. This implies

$$\frac{\partial \psi_m}{\partial \tau_i}(\tau) = Y_i(\Phi(m, \tau)) \quad \forall i = 1, ..., n.$$

We get that the Jacobian matrix has all independent columns, and hence ψ_m is a local diffeomorphism by inverse function theorem.

The manifold M is connected, hence it is even path connected. Call $\alpha : [0, 1] \to M$ the smooth path connecting two points $m = \alpha(0)$ and $m' = \alpha(1)$ of M. Proving surjectivity is equivalent to proving the existence of a $\bar{\boldsymbol{\tau}}$ such that $\psi_m(\bar{\boldsymbol{\tau}}) = m'$. For any point $n \in \alpha$, where by α we are indicating the image of the curve, there exists an open neighbourhood N_n such that $\psi_n|_{N_n}$ is a diffeomorphism. This collection of open sets is an open covering of the compact set represented by the image of the curve α. By compactness, we can extract a finite set of points $\{n_1, ..., n_k\}$ such that $\{N_{n_1}, ..., N_{n_k}\}$ cover α. Taking care that the adjacent neighbourhoods have non-trivial intersection, we can transition from n_i to n_{i+1} thanks to some $\boldsymbol{\tau}_i \in \mathbb{R}^n$. We can now set $\boldsymbol{\tau} = \boldsymbol{\tau}_1 + ... + \boldsymbol{\tau}_k$ and get

$$\psi_{\boldsymbol{\tau}}(m) = m'.$$

This implies that ψ_m is surjective for any m in M and, equivalently, that the group action Φ is transitive.

□

Lemma 2. *Each point $m \in M$ has the same isotropy group with respect to the action Φ of $(\mathbb{R}^n, +)$ on M. Moreover, this group is discrete.*

Proof. We have just proven that ψ_m is a local diffeomorphism, so in particular it is injective in a neighbourhood of any point $m \in M$. This means that any $\tau \in G_m$ is an isolated point, i.e. the isotropy group is discrete. Moreover, all the isotropy subgroups coincide since the action Φ is transitive. Indeed, for any pair of points $m, n \in M$ we have

$$n = \Phi(\boldsymbol{t}, m) = \psi_{\boldsymbol{t}}(m) \text{ for some } \boldsymbol{t} \in \mathbb{R}^n.$$

Hence, if $\boldsymbol{\tau} \in G_m$, then

$$\Phi(\boldsymbol{\tau}, n) = \psi_{\boldsymbol{\tau}} \circ \psi_{\boldsymbol{t}}(m) = \psi_{\boldsymbol{t}} \circ \psi_{\boldsymbol{\tau}}(m) = \psi_{\boldsymbol{t}}(m) = n$$

and $\boldsymbol{\tau} \in G_n$. □

Another property of the isotropy group is that it is not trivial, otherwise the map ψ_m would be injective for any m and this is impossible since M is compact by assumption. As every discrete subgroup of $\mathbb{R}^n$, $G_m = G$ is a $k-$dimensional lattice of $\mathbb{R}^n$ $(1 < k \leq n)$ and it can be written as

$$G = \{\boldsymbol{\tau} = \sum_{i=1}^{k} \boldsymbol{\nu}_i c_i : \ c_i \in \mathbb{Z}, \ \boldsymbol{\nu}_i \in \mathbb{R}^n\}.$$

The set of vectors $\{\boldsymbol{\nu}_1, ..., \boldsymbol{\nu}_k\}$ is called *basis of the lattice*. This is not unique in general. Every $k-$dimensional lattice of $\mathbb{R}^n$ is isomorphic to the standard lattice with basis $e_1, ..., e_k$, where the e_i's are the vectors of the canonical basis of $\mathbb{R}^n$. To see this fact, we can complete the basis $\{\boldsymbol{\nu}_1, ..., \boldsymbol{\nu}_k\}$ to a basis of $\mathbb{R}^n$ adding $(n-k)$ linearly independent vectors $\boldsymbol{\nu}_{k+1}, ..., \boldsymbol{\nu}_n$. We can then define a coordinate transformation via the matrix L:

$$L_{ij} = \boldsymbol{\nu}_j \cdot e_i.$$

Now, up to this change of coordinates, we can suppose $G = \{\sum_{i=1}^{k} c_i e_i : \ c_i \in \mathbb{Z}\}$. Let's consider the quotient space $\mathbb{R}^n/G$, where each pair of points $x, y \in \mathbb{R}^n$ with $x - y \in \mathbb{Z}^k \times \{0 \in \mathbb{R}^{n-k}\}$ belong to the same equivalence class. This implies that $\mathbb{R}^n/G \simeq \mathbb{T}^k \times \mathbb{R}^{n-k}$.

The surjective map $\psi_m : \mathbb{R}^n \to M$ induces the global diffeomorphism

$$\hat{\psi}_m : \mathbb{R}^n/G \simeq \mathbb{T}^k \times \mathbb{R}^{n-k} \to M$$

defined in such a way that $\forall t \in \mathbb{R}^n$,

$$\psi_m(t) = \hat{\psi}_m([t]).$$

This can be done since ψ_m is constant on the fibers of $\pi : \mathbb{R}^n \to \mathbb{T}^k \times \mathbb{R}^{n-k}$. Indeed, if $\psi_m(\boldsymbol{t}) = \psi_m(\boldsymbol{\tau})$, then we get

$$\psi_m(\boldsymbol{t}) = \Phi(\boldsymbol{t}, m) = \Phi(\boldsymbol{\tau}, m) = \psi_m(\boldsymbol{\tau}), \quad \boldsymbol{t}, \boldsymbol{\tau} \in \mathbb{R}^k,$$

which when set as the argument of $\Phi_{-\boldsymbol{t}}$ brings to

$$m = \Phi(-\boldsymbol{t}, \Phi(\boldsymbol{\tau}, m)) = \Phi(\boldsymbol{\tau} - \boldsymbol{t}, m),$$

which means $\boldsymbol{t} - \boldsymbol{\tau} \in G$ (i.e. $[\boldsymbol{t}] = [\boldsymbol{\tau}]$). $\hat{\psi}_m$ is surjective since ψ_m is surjective. To show injectivity, we need to show $\hat{\psi}_m([\boldsymbol{x}]) = \hat{\psi}_m([\boldsymbol{y}])$ if and only if $\boldsymbol{x} \sim \boldsymbol{y}$. This equality implies $\psi_m(\boldsymbol{x}) = \psi_m(\boldsymbol{y})$, and hence $\boldsymbol{x} - \boldsymbol{y} \in G$. So $\hat{\psi}_m$ is an isomorphism. To prove it is a global diffeomorphism, is enough to prove it is a local diffeomorphism. Both π and ψ_m are local diffeomorphisms (precisely, the first k components of π being a covering map of $\mathbb{T}^k$ can be turned into a map with constant rank k endowing $\mathbb{R}^k$ with a natural differentiable structure, while the remaining components are the identity map), so since $\psi_m = \hat{\psi}_m \circ \pi$, we can derive the desired result.

Since M is compact and compactness is preserved by diffeomorphisms, even $\mathbb{R}^n/G$ is compact. From this follows $n = k$ and we can conclude that M is diffeomorphic to a $n-$dimensional torus.

□

This proposition implies that all the invariant fibers of F are diffeomorphic to $k-$dimensional tori and the action $\Phi_t := \Phi_{t_1}^{Y_1} \circ ...\Phi_{t_k}^{Y_k}$ generated by the symmetries of X is transitive. We now need to show that, on each invariant torus, the dynamics can be conjugated to a linear flow. Consider the vector fields

$$\bar{Y}_i = \sum_{j=1}^{k} L_{ji} Y_j.$$

The matrix L is invertible, so these new vector fields are independent and they commute with X since they are a linear combination with constant coefficients of the Y_js. Moreover, by bilinearity of the Jacobi-Lie bracket, they pairwise commute:

$$[\bar{Y}_m, \bar{Y}_n] = \sum_{j,h=1}^{k} L_{jm} L_{hn} [Y_j, Y_h] = 0.$$

They preserve the first integrals of X, in fact

$$\mathcal{L}_{\bar{Y}_m} f_h = df_h \Big(\sum_{j=1}^{k} L_{jm} Y_j \Big) = \sum_{j=1}^{k} L_{ji} df_h(Y_j) = 0, \quad \forall h = 1, ..., n-k.$$

This means that they are k commuting vector fields tangent to the invariant torus N. They hence define an action of $\mathbb{R}^k$ on it.

Claim : These fields are the generators of the action Φ, when the period lattice is expressed with respect to the canonical basis:

$$\hat{\Phi}(t,m) := \Phi(Lt,m) = \Phi^{\bar{Y}_1}_{t_1} \circ ... \circ \Phi^{\bar{Y}_k}_{t_k}(m).$$

Indeed, suppose that $t \in G \in \{\sum_{i=1}^n c_i \boldsymbol{e}_i \,:\, c_i \in \mathbb{Z}\}$, then we can apply the change of coordinates $L : \mathbb{R}^k \to \mathbb{R}^k$:

$$\begin{aligned}
\hat{\Phi}_t = \Phi_{Lt} = \Phi_{(L_{1j}t_j,...,L_{kj}t_j)} = \\
&= \Phi^{Y_1}_{L_{1j}t_j} \circ ... \circ \Phi^{Y_k}_{L_{kj}t_j} = \Phi_1^{\sum_{j=1}^k t_j L_{1j} Y_1} \circ ... \circ \Phi_1^{\sum_{j=1}^k t_j L_{kj} Y_k} = \\
&= \Phi_1^{\sum_{j=1}^k t_j \sum_{i=1}^k L_{ij} Y_i} = \Phi_1^{\sum_{j=1}^k t_j \bar{Y}_j} = \Phi^{\bar{Y}_1}_{t_1} \circ ... \Phi^{\bar{Y}_k}_{t_k}.
\end{aligned}$$

Let's now show that all these fields have only periodic orbits with period 1. In the canonical basis, the isotropy group G can be identified with $\mathbb{Z}^k$ so

$$\Phi(\boldsymbol{t},m) = \Phi(\boldsymbol{\tau},m)$$

if and only if $\boldsymbol{t} - \boldsymbol{\tau} \in \mathbb{Z}^k$. Consider

$$\boldsymbol{t} = (t_1, 0, ..., 0) \text{ and } \boldsymbol{\tau} = (\tau_1, 0, ..., 0).$$

This implies $t_1 - \tau_1 \in \mathbb{Z}$, which means that all the orbits of all these vector fields are $1-$periodic. By Proposition 20, we get that $\hat{\Phi}$ is an action of $\mathbb{T}^k$ on M. Moreover, the fields $\bar{Y}_1, ..., \bar{Y}_k$ form a basis of the tangent space to each invariant torus $N = \{F = \boldsymbol{f} \in \mathbb{R}^{n-k}\} \subset M$. Hence

$$X|_N = \sum_{i=1}^k \omega_i \bar{Y}_i,$$

for some $\omega_i : N \to \mathbb{R}$, $i = 1, ..., k$. The ω_is are constant on each invariant torus. Indeed,

$$0 = [\bar{Y}_k, X] = \sum_{i=1}^k \omega_i(\boldsymbol{f}, \alpha_1, ..., \alpha_k)[\bar{Y}_k, \bar{Y}_i] + \sum_{i=1}^k \mathcal{L}_{\bar{Y}_k}(\omega_i)\bar{Y}_i = \sum_{i=1}^k \mathcal{L}_{\bar{Y}_k}(\omega_i)\bar{Y}_i$$

and by linear independence of the $\bar{Y}_i$s this implies $\omega_i = c_i \in \mathbb{R}$ for any $i = 1, ..., k$. From this follows the desired local result:

$$X|_N = \sum_{i=1}^k c_i \frac{\partial}{\partial \alpha_i},$$

where we set $\bar{Y}_i = \partial/\partial\alpha_i$ since they are all $1-$periodic so the coordinates describing the motion along their integral curves can be seen as angular coordinates.

We now need to show how to extend the construction done above to a tubular neighbourhood of any invariant level set of the fibration F. Consider a point $m \in M$ and a small enough open neighbourhood $P \subset M$ such that the map

$$\mathcal{C} = (F, y) : P \to \mathbb{R}^{n-k} \times \mathbb{R}^k$$

is a diffeomorphism onto its image. This map defines a coordinate system on P and, up to considering a subset of P, we can suppose there is the product structure: $\mathcal{C}(P) = \mathcal{F} \times \mathcal{Y}$, where $F(P) = \mathcal{F}$ and $y(P) = \mathcal{Y}$. Let $(\bar{\boldsymbol{f}}, \bar{\boldsymbol{y}}) = \mathcal{C}(m)$ and consider the section $\sigma : \mathcal{F} \to P$ defined as $\sigma(f) := \mathcal{C}^{-1}(f, \bar{y})$. $\sigma(\mathcal{F}) \subset M$ is a $(n-k)$-dimensional embedded submanifold and it intersects in exactly one point each level set of F with non-trivial intersection with P. To proceed, we now consider the same action of $\mathbb{R}^k$ as before, but this time we extend it to the whole set $F^{-1}(\mathcal{F})$. Recalling that for each $f \in \mathcal{F}$ we have an associated invariant torus, we see that this action is defined on a collection of invariant tori. The action reads

$$\Phi : \mathbb{R}^k \times F^{-1}(\mathcal{F}) \to F^{-1}(\mathcal{F}),\ (\boldsymbol{\tau}, \sigma(f)) \to \Phi(\boldsymbol{\tau}, \sigma(f))$$

and is generated by the commuting vector fields $Y_1, ..., Y_k$.

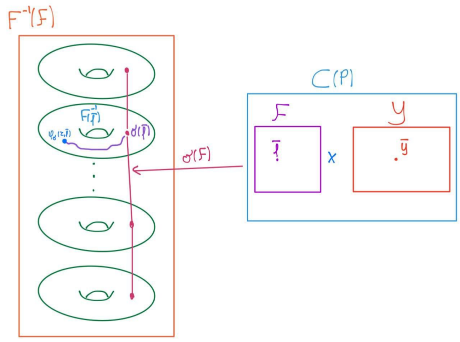


Figure 3.1: Visual description of these mappings

The induced map

$$\varphi_\sigma : \mathcal{F} \times \mathbb{R}^k \to F^{-1}(\mathcal{F}),\ \varphi_\sigma(\boldsymbol{f}, \boldsymbol{\tau}) = \Phi(\boldsymbol{\tau}, \sigma(\boldsymbol{f}))$$

is well defined, surjective and differentiable since we have proven these properties on each invariant torus. Indeed, for each fixed $f \in \mathcal{F}$ the map

$$\varphi_{f,\sigma} : \mathbb{R}^k \to F^{-1}(f)$$

has these properties and, changing the value of $f \in \mathcal{F}$, we can select each and every invariant torus in $F^{-1}(\mathcal{F})$ by construction. Let's now compute the Jacobian matrix of φ_σ. With the same computations done in the proof of Proposition 21, we have

$$\frac{\partial \varphi_\sigma}{\partial \tau_k}(\bar{\boldsymbol{f}}, 0) = Y_k(\Phi(0, \sigma(\bar{\boldsymbol{f}}))) = Y_k(\sigma(\bar{\boldsymbol{f}})).$$

We can even compute

$$\frac{\partial \sigma}{\partial f_i}(\bar{\boldsymbol{f}}) = \frac{\partial \mathcal{C}^{-1}}{\partial f_i}(\bar{\boldsymbol{f}}, 0)$$

and since $\mathcal{C}$ is a diffeomorphism, this implies that $\partial_{f_i}\varphi_\sigma$ are all independent. At each point $m \in \sigma(\mathcal{F})$, by transversality, we have

$$T_m M = T_m \sigma(\mathcal{F}) \oplus \langle Y_1(m), ..., Y_k(m) \rangle.$$

This implies that all the columns of the Jacobian matrix are independent and hence, by inverse function theorem, φ_σ is a local diffeomorphism at $(\bar{\boldsymbol{f}}, 0)$. Let's now denote with $\mathcal{A} \times \mathcal{B}$ an open neighbourhood of $(\bar{\boldsymbol{f}}, 0)$ such that

$$\varphi_\sigma : \mathcal{A} \times \mathcal{B} \to \varphi_\sigma(\mathcal{A} \times \mathcal{B})$$

is a diffeomorphism. We can notice that even $\Phi_{\boldsymbol{\tau}} \circ \varphi_\sigma$ is a diffeomorphism on $\mathcal{A} \times \mathcal{B}$ for any $\tau \in \mathbb{R}^{n-k}$. This allows us to naturally define a diffeomorphism on every product space with structure $\mathcal{A} \times (\boldsymbol{\tau} + \mathcal{B})$ since $\forall t \in (\mathcal{B} + \tau)$, we have $(t - \tau) \in \mathcal{B}$ and we can set

$$\varphi_\sigma : \mathcal{A} \times (\tau + \mathcal{B}) \to \varphi_\sigma(\mathcal{A} \times (\tau + \mathcal{B})),\ \varphi_\sigma(f, t) := \Phi_\tau \circ \varphi_\sigma(f, t - \tau).$$

We have already proven that on each invariant torus, the isotropy group is one and the same. We denote it with $G_{\sigma(f)}$. The important thing to check is if this group varies in a differentiable way passing from an invariant torus to those in its tubular neighbourhood. We define the map

$$G : \mathcal{F} \to \mathbb{R}^k,\ f \to G_{\sigma(f)}$$

describing how the period lattice (and even matrix) changes from torus to torus. Let's recall that $t \in G_{\sigma(f)}$ if and only if $\varphi_\sigma(t, f) = \Phi(t, \sigma(f)) = \sigma(f)$. We see that:

$$\frac{\partial \Phi}{\partial t_i}(\boldsymbol{t}, \sigma(f)) = Y_i(\Phi(\boldsymbol{t}, \sigma(f))).$$

This means the relation $\Phi(t, \sigma(f)) - \sigma(f) = 0$ can be turned into a differentiable relation of the form $t = G(f) = G_{\sigma(f)}$ by implicit function theorem. Namely, there exists a set of functions

$$\mathcal{A} \ni f \to (u_1(f), ..., u_k(f)) \in \mathbb{R}^{k \times k}$$

which are differentiable defining the basis of the period lattice on each invariant torus $F^{-1}(f)$. We can define the change of basis as follows:

$$L_{ij}(f) = u_j(f) \cdot e_i.$$

By construction, this depends in a differentiable way on $f \in \mathcal{A}$. We can now conclude as in the local version of the Theorem, defining the fields

$$\bar{Y}_i = \sum_{j=1}^{k} L_{ji}(f) Y_j.$$

For any fixed $f \in \mathcal{A}$ these vector fields pairwise commute, conserve the first integrals of X, commute with X, and have all $1-$periodic orbits. This implies that the same holds in this case. We can consider the diffeomorphism

$$\hat{\psi}_\sigma : \mathcal{F} \times \mathbb{T}^k \to F^{-1}(\mathcal{F})$$

which is generated by the fields $\bar{Y}_i$, namely

$$\hat{\psi}_\sigma(f, \boldsymbol{\alpha} \mod 1) := \Phi^{\bar{Y}_1}_{\alpha_1} \circ ... \circ \Phi^{\bar{Y}_k}_{\alpha_k}(\sigma(f)).$$

The restriction of X on $F^{-1}(\mathcal{A})$ is given by

$$X|_{F^{-1}(\mathcal{A})} = \sum_{j=1}^{k} \omega_j(f, \boldsymbol{\alpha})\bar{Y}_j,$$

and since $[X|_{F^{-1}(A)}, \bar{Y}_h] = 0$ for any $h = 1, ..., k$, we get:

$$0 = [\bar{Y}_h, X|_{F^{-1}(\mathcal{A})}] = \sum_{j=1}^{k} \omega_j(f, \alpha)[\bar{Y}_j, \bar{Y}_h] + \sum_{j=1}^{k} \mathcal{L}_{\bar{Y}_h}(\omega_j)\bar{Y}_j = \sum_{j=1}^{k} \mathcal{L}_{\bar{Y}_h}(\omega_j)\bar{Y}_j.$$

By linear independence of the $\bar{Y}_h$s, we get

$$\mathcal{L}_{\bar{Y}_h}(\omega_i) = 0, \quad \forall h, i = 1, ..., k.$$

Recalling that

$$\bar{Y}_h = \frac{\partial}{\partial \alpha_h},$$

we get

$$\mathcal{L}_{\bar{Y}_h}\omega_i = \frac{\partial \omega_i}{\partial \alpha_h}(f, \alpha) = 0, \quad \forall h = 1, ..., k, \, \forall i = 1, ..., k.$$

Namely, these functions are constant on each invariant torus: $\omega_i = \omega_i(f)$. This allows us to conclude the proof and get

$$X|_{F^{-1}(\mathcal{A})} = \sum_{j=1}^{k} \omega_j(f) \frac{\partial}{\partial \alpha_j}.$$

□

3.5 Torus actions and Bogoyavlensky's Theorem

In this Section we consider integrable systems of type (p, q) defined on a smooth $n-$dimensional manifold M ($p + q = n$, as defined in Section 3.2). Most of the ideas and results presented in this last Section come from [16].

Theorem 9 (Non-Hamiltonian integrability)**.** *Consider $(Y_1, ..., Y_p, F_1, ..., F_q)$, an integrable system of type (p, q), defined on the $(p + q)$-dimensional manifold M. Suppose the fibers of the submersion $F = (F_1, ..., F_q) : M \to \mathbb{R}^q$ to be regular, connected and compact. Then, any level set $N \subset M$ of F admits a tubular neighbourhood $\mathcal{U}(N) \simeq N \times B^q \subset M$, B^q open ball of $\mathbb{R}^q$, and a $\mathbb{T}^p$-action $\Phi : \mathbb{T}^p \times \mathcal{U}(N) \to \mathcal{U}(N)$ which is free and preserves the system (i.e. the dynamical symmetries and the first integrals), whose orbits are the level sets of $F = (F_1, ..., F_q)$. Moreover, there exists a system of coordinates $(\alpha_1, ..., \alpha_p, a_1, ..., a_q)$ identifying $\mathcal{U}(N)$ with $\mathbb{T}^p \times B^q$. Furthermore, in this coordinate system, the vector fields defining the dynamics read*

$$Y_i = \sum_{j=1}^{p} b_{ij}(a_1, ..., a_q) \frac{\partial}{\partial \alpha_j} \quad i = 1, ..., p. \tag{3.1}$$

We refer to $(\alpha_1, ..., \alpha_p, a_1, ..., a_q)$ as *Liouville coordinates.* As seen in the Section 3.4, these regular level sets are all diffeomorphic to $p-$dimensional tori and we call them *Liouville tori.*

Proof. To define the system of Liouville coordinates, we show that F is a locally-trivial fibration. Indeed, we analyze two equivalent proofs to this problem. The former is the most classical and standard one, while the latter is reported because it allows to build in a constructive way the diffeomorphism and it even explains why we often refer to the open neighbourhood of N as a *tubular neighbourhood.*

1. The map $F : \mathcal{U}(N) \to F(\mathcal{U}(N)) := \mathcal{F}$ is a surjective submersion, and it is proper since its level sets are compact. Then, by Ehresmann's Theorem, it is a locally trivial-fibration. This precisely means that if $N = F^{-1}(f)$, then there exists an open neighbourhood $U_f \subset \mathbb{R}^q$ of f such that

$$\varphi : F^{-1}(U_f) \to F^{-1}(f) \times U_f \simeq \mathbb{T}^p \times U_f$$

is a diffeomorphism. Therefore, we define the tubular neighbourhood of N as

$$\mathcal{U}(N) = F^{-1}(U_f).$$

2. The idea behind this second approach comes from [12]. Consider a Riemannian metric g on M. Define the exponential map induced by the metric at a point $n \in N$

$$exp_n : T_nM \to M,\ v \to \gamma_v(1),$$

where $\gamma_v : \mathbb{R} \to M$ is the unique geodesic curve satisfying

$$\gamma_v(0) = n \text{ and } \gamma_v'(0) = v.$$

This map sends an open neighbourhood of the zero vector of T_nM to an open neighbourhood of n in M and is a local diffeomorphism. The normal bundle of the Liouville torus N is

$$Norm\, N := \{(x, v) \in TM : x \in N,\ v \in\ Norm_x N\},$$

where

$$Norm_x N := \{v \in T_xM : g(v, w) = 0, \forall w \in T_xN\}$$

and $\pi : Norm\, N \to N$ is the canonical projection $(x, v) \to x$. N can be identified with the zero-section of its normal bundle:

$$\sigma : N \to Norm\, N,\ x \to (x, 0) \in N \times \{0\}.$$

We can now conclude thanks to the exponential map. Indeed, consider an arbitrary point $m \in N$. Thanks to the exponential map, we can identify a neighbourhood $U_m \subset M$ with a neighbourhood $V_m \subset Norm\, N$. Precisely, geodesics are curves realizing the distances between points, and a point $(x, v) \in Norm\, N$ has as the nearest point on N exactly x. We then consider the map

$$\Phi_m : V_m \to U_m,\ (x, v) \to \exp_x(v)$$

with V_m and U_m small enough so that Φ_m is a diffeomorphism. Hence, we can associate each $x \in N$ to all the points in the neighbourhood $\mathcal{U}(N)$ of the zero-section of the form (x, v). This construction can be repeated with any point in N and, by compactness, we can cover N with finitely many open sets

$$N \subset \cup_{i=1}^{k} U_{m_i} = \mathcal{U} \subset M$$

where these diffeomorphisms are well defined. Furthermore, they are compatible one with the others, since they coincide with the exponential map. We can therefore extend these local maps to a global one identifying $\mathcal{U}$ with a tubular neighbourhood $\mathcal{U}(N) \simeq N \times B^q$, where B^q is an open ball of $\mathbb{R}^q$.

The existence of the free torus action comes from the same approach followed in the proof of Bogoyavlensky's integrability Theorem. Indeed, the vector fields $Y_1, ..., Y_p$ define an action of $\mathbb{R}^p$ on $\mathcal{U}(N)$. Each level set of $F = (F_1, ..., F_q)$ is a connected component of an integral manifold of the tangent distribution $\mathcal{D} = span\{Y_1, ..., Y_p\}$, and is hence a $p-$dimensional torus when compact. The period lattice on each torus is unique and we can uniform those of the tori fibrating $\mathcal{U}(N)$ with the change of basis

$$(u_1(f), ..., u_p(f)) \xrightarrow{L^{-1}} (e_1, ..., e_p),$$

where $(e_1, ..., e_p)$ is the canonical basis of $\mathbb{R}^p$. We define then another action of $\mathbb{R}^p$ on $\mathcal{U}(N)$ as

$$\hat{\Phi}(t, m) := \Phi(Lt, m)$$

which induces an action of $\mathbb{T}^p$ over $\mathcal{U}(N)$ since its generators are all $1-$periodic vector fields. Now call $W_1, ..., W_p$ the generators of the torus action and define as angular coordinates the parameters moving along their integral curves. Namely,

$$W_i = \frac{\partial}{\partial \alpha_i}, \quad i = 1, ..., p.$$

To obtain the expression in (3.1), the procedure is exactly the same seen in Bogoyavlensky's Theorem. We just need to fix a vector field among $Y_1, ..., Y_p$, i.e. set $X = Y_i$, and then repeat the procedure with the others. □

We call the torus action generated by the vector fields $W_1, ..., W_p$ *Liouville torus action*. This action is very important both from the practical and conceptual point of view. The following Theorem, gives one of the main reasons behind this relevance.

Theorem 10 (Fundamental conservation property)**.** *Consider a Liouville torus N of a (p, q)-integrable system $(Y_1, ..., Y_p, F_1, ..., F_q)$ defined on a smooth manifold M of dimension $(p + q)$. Moreover, assume $\mathcal{G} \in \Gamma(\otimes^h TM \otimes^k T^*M)$ is an invariant tensor field for the system, i.e. $\mathcal{L}_{Y_i}\mathcal{G} = 0$ for any $i = 1, ..., p$. Then, the Liouville torus action*

$$\Phi : \mathbb{T}^p \times \mathcal{U}(N) \to \mathcal{U}(N)$$

preserves this tensor field, where $\mathcal{U}(N)$ is a tubular neighbourhood of N endowed with a system of Liouville coordinates.

Proof. Consider a system of Liouville coordinates $(\alpha_1, ..., \alpha_p, a_1, ..., a_q)$ on $\mathcal{U}(N)$, where we suppose the torus action Φ is generated by

$$W_i := \frac{\partial}{\partial \alpha_i}, \quad i = 1, ..., p.$$

Fix a natural number $s \in [0, h+k]$ and define the space

$$\mathcal{T}_s \subset \Gamma(\otimes^h TM \otimes^k T^*M)$$

as the one made by elements of the type

$$\epsilon = \frac{\partial}{\partial \alpha_{i_1}} \otimes ... \otimes \frac{\partial}{\partial \alpha_{i_b}} \otimes \frac{\partial}{\partial a_{j_1}} \otimes ... \otimes \frac{\partial}{\partial a_{j_c}} \otimes d\alpha_{i'_1} \otimes d\alpha_{i'_u} \otimes da_{j'_1} \otimes da_{j'_v}$$

where $b + c = h$, $u + v = k$, $c + u \leq s$. Notice that since with respect to the Liouville coordinates

$$Y_i = \sum_{h=1}^{p} b_{ih}(a) \frac{\partial}{\partial \alpha_h},$$

follows

$$\mathcal{L}_{Y_i} \frac{\partial}{\partial a_k} = \Big[\sum_{h=1}^{p} b_{ih}(a) \frac{\partial}{\partial \alpha_h}, \frac{\partial}{\partial a_k} \Big] = - \sum_{h=1}^{p} \frac{\partial b_{ih}(a)}{\partial a_k} \frac{\partial}{\partial \alpha_h} \in span\{W_1, ..., W_p\},$$

$$\mathcal{L}_{Y_i} d\alpha_k = d\mathcal{L}_{Y_i} \alpha_k = d(b_{ik}(a)) = \sum_{j=1}^{q} \frac{\partial b_{ik}(a)}{\partial a_j} da_j \in span\{da_1, ..., da_q\},$$

$$\mathcal{L}_{Y_i} \frac{\partial}{\partial \alpha_k} = \mathcal{L}_{Y_i} da_k = 0.$$

This means that if ϵ generated $\mathcal{T}_s$, then $(\mathcal{L}_{Y_i} \epsilon) \in \mathcal{T}_{s-1}$. Furthermore, by construction we see that

$$\mathcal{T}_{-1} \subset \mathcal{T}_0 \subset ... \subset \mathcal{T}_{s-1} \subset \mathcal{T}_s \subset ... \subset \mathcal{T}_{h+k} = \Gamma(\otimes^h TM \otimes^k T^*M),$$

where $\mathcal{T}_{-1} := \{0\}$. By assumption, $\mathcal{L}_{Y_i} \mathcal{G} = 0$ for any $i = 1, ..., p$. Defining $\bar{\mathcal{G}}$ the average[1] of this tensor with respect to the torus action Φ on $\mathcal{U}(N)$, we get that even $\mathcal{L}_{Y_i} \bar{\mathcal{G}} = 0$ for any $i = 1, .., p$. Precisely, set

$$\bar{\mathcal{G}}_m(\mu_1, ..., \mu_{b+c}, Z_1, ..., Z_{u+v}) := \int_{\mathbb{T}^p} (\Phi^*_\alpha \mathcal{G})_m(\mu_1, ..., \mu_{b+c}, Z_1, ..., Z_{u+v}) \; d\alpha_1 \wedge ... \wedge d\alpha_p$$

where $m \in \mathcal{U}(N)$, $\mu_i \in T^*(\mathcal{U}(N))$, $Z_j \in T(\mathcal{U}(N))$. This shows why $\mathcal{L}_{Y_i} \bar{\mathcal{G}} = 0$, since the Lie derivative goes inside the sign of integral and commutes with the pull-back because W_i commutes with the vector fields generating Φ.

We now set $\hat{\mathcal{G}} = \mathcal{G} - \bar{\mathcal{G}}$ and prove $\hat{\mathcal{G}} \equiv 0$ by induction. Precisely, $\hat{\mathcal{G}} \in \mathcal{T}_{h+k}$ and, assuming it belongs to a certain $\mathcal{T}_s$, we now prove it belongs even to $\mathcal{T}_{s-1}$. Suppose, by contradiction, that $\hat{\mathcal{G}}$ has a term $\rho \in \mathcal{T}_s \setminus \mathcal{T}_{s-1}$ of the form

$$\rho = \varphi \frac{\partial}{\partial \alpha_{i_1}} \otimes ... \otimes \frac{\partial}{\partial \alpha_{i_b}} \otimes \frac{\partial}{\partial a_{j_1}} \otimes ... \otimes \frac{\partial}{\partial a_{j_c}} \otimes d\theta_{i'_1} \otimes d\theta_{i'_u} \otimes dz_{j'_1} \otimes dz_{j'_v} = \varphi \, \epsilon.$$

[1] The averaging technique is useful to generate other invariant tensors as explained in [18], Chapter 9, Introduction to Lie Groups

We have already seen that for any $i = 1, ..., p$,

$$\mathcal{L}_{Y_i}\rho = Y_i(\varphi)\,\epsilon + \Gamma, \quad \Gamma \in \mathcal{T}_{s-1}.$$

Moreover, by linearity of the Lie derivative, $\mathcal{L}_{Y_i}\hat{\mathcal{G}} = 0$ implies necessarily $Y_i(\varphi) = 0$ and hence the coefficient of this monomial term is necessarily constant on each Liouville torus. Furthermore, by construction, $\hat{\mathcal{G}}$ has zero average on each torus ($\bar{\hat{\mathcal{G}}} = \bar{\mathcal{G}} - \bar{\bar{\mathcal{G}}} = 0$) and this implies that the average on each torus of φ is 0 too (i.e. $\varphi|_N \equiv 0$). This is a contradiction since if $\varphi = 0$ then $\rho \in \mathcal{T}_{s-1}$. Hence follows that it is not possible to have a monomial term in $\hat{\mathcal{G}}$ which belongs to $\mathcal{T}_s \setminus \mathcal{T}_{s-1}$. By induction, we get that the same holds when $s = 0$ and this implies $\hat{\mathcal{G}} \in \mathcal{T}_{-1} = \{0\}$. We can now say that $\mathcal{G} = \bar{\mathcal{G}}$, which concludes the proof since it means that $\mathcal{G}$ is invariant with respect to the torus action Φ (namely, $\bar{\bar{\mathcal{G}}} = \bar{\mathcal{G}} = \mathcal{G}$).

$\square$

Chapter 4

Euler-Jacobi Theorem

Some dynamical systems defined on a $n-$dimensional smooth manifold M, which do not have a natural Hamiltonian structure, can still be integrable even when the hypotheses of Bogoyavlensky's Theorem do not hold. As we are going to see now, when there are $n-2$ functionally independent first integrals and an invariant volume form, then we have a sort of integrability. With these available quantities (plus some technical assumptions on the phase space and on the analyzed vector field), a vector field is conjugated to a linear flow on each invariant torus (up to a time reparametrization).

4.1 Statement of the Theorem

In this section, we state the Euler-Jacobi Theorem, giving some context and reference for mechanical systems which are integrable in this sense.

Theorem 11 (Euler-Jacobi)**.** *Consider a smooth vector field V on an orientable and smooth manifold M of dimension n. Assume*

$$F = (f_1, ..., f_{n-2}) : M \to \mathbb{R}^{n-2}$$

is a submersion whose components are functionally independent first integrals of V. Moreover, let μ be an invariant volume form for the system (i.e. $\mathcal{L}_V \mu = 0$). Consider a regular invariant level set of F

$$N = \{f_1 = c_1, ..., f_{n-2} = c_{n-2}\}$$

and suppose it to be compact, connected and that V does never vanish on it. Then N and all the nearby integral surfaces are diffeomorphic to $\mathbb{T}^2$. Furthermore, in some neighborhood of N there exist local coordinates $f_1, ..., f_{n-2}, x_{n-1}, x_n$ such that x_{n-1}, x_n are angular coordinates on the tori and in these coordinates the vector field takes the form

$$\begin{cases} \dot{f}_i = 0, \quad i = 1, ..., n-2 \\ \dot{x}_j = \frac{\lambda_j(f)}{\Phi(f, x_{n-1}, x_n)}, \quad j = n-1, n. \end{cases} \tag{4.1}$$

Moreover, if the original vector field, its integrals of motion and the invariant volume form are real analytic, then so are also the functions x_j, Φ and λ_j with $j = n-1, n$.

As we can see from the expression (4.1), the scalar function Φ motivates why at the beginning of this Chapter we have said this Theorem gives a *sort* of integrability result. Indeed, here emerges that the vector field conjugated by the local system of coordinates $(f_1, ..., f_{n-2}, x_{n-1}, x_n)$ to a quasi-periodic flow on the invariant tori is not V, but the rescaling ΦV.

A classical example of system which is integrable in the sense of Euler-Jacobi is the Chaplygin sphere, a non-holonomic mechanical system [23],[24],[25].

4.2 Proof of the Theorem

In this section we give a detailed proof of Euler-Jacobi Theorem. Since it is quite long, we split it into various steps.

Proof. First of all let's notice that since F defines a submersion, the invariant level set N is a $2-$dimensional embedded submanifold of M. We start with a classical result in differential geometry:

Proposition 22 (Corollary to Poincaré-Hopf Theorem)**.** *If a 2-dimensional smooth, connected and compact manifold S admits a smooth nowhere vanishing tangent vector field, then it is homeomorphic to $\mathbb{T}^2$.*

The Poincaré-Hopf Theorem is stated and presented in Appendix A.

Proof. The proof of this theorem is based on the Poincaré-Hopf Theorem on the Euler characteristic of a surface. Having a nowhere vanishing vector field on a compact, connected and orientable surface implies that its Euler characteristic is $\chi(S) = 0$. This value is a topological invariant. Hence we can conclude by the classification of compact and connected surfaces that S is homeomorphic to $\mathbb{T}^2$. □

This first result allows us to use two angular coordinates to describe the dynamics on each regular invariant level set of the submersion. The second step is to show that the restriction of the invariant volume form μ on the invariant set N is still an invariant form for $X = V|_N$.

Proposition 23. *Suppose that a vector field $V \in \mathfrak{X}(M)$ on a $n-$dimensional manifold M has an invariant volume form μ and $(n-2)$ first integrals $f_1, ..., f_{n-2}$ defining the following invariant surface*

$$N = \{f_1 = c_1, ..., f_{n-2} = c_{n-2}\} = \{F = \boldsymbol{c}\}$$

which is supposed to be regular, i.e. $df_1 \wedge ... \wedge df_{n-2} \neq 0$ on N. Then the vector field $X = V|_N$ admits an invariant measure ω defined by a $2-$form as

$$\omega \wedge df_1 \wedge ... \wedge df_{n-2} = \mu.$$

Proof. Consider a local system of coordinates with

$$f_1 = x_1, ..., f_{n-2} = x_{n-2}.$$

The volume form μ can be written as

$$\mu = \rho(x_1, ..., x_n)\, dx^1 \wedge ... \wedge dx^n,$$

so its restriction ω on N is of the form

$$\omega = \mu|_N = \rho(c_1, ..., c_{n-2}, x_{n-1}, x_n)dx^{n-1} \wedge dx^n = \tilde{\rho}_{\mathbf{c}}(x_{n-1}, x_n)dx^{n-1} \wedge dx^n.$$

Let's now denote with $V^1, ..., V^n$ the components of the vector field V with respect to this coordinate system. Since the f_is are first integrals, we know that for any $i = 1, ..., n-2$

$$V^i = dx^i(V) = df_i(V) = V(f_i) = \mathcal{L}_V f_i = 0.$$

Define $\alpha = df_1 \wedge ... \wedge df_{n-2} \in \Lambda^{n-2}(M)$. We see that

$$L_V \alpha = \sum_{i=1}^{n-2} df_1 \wedge ... \wedge df_{i-1} \wedge d(\mathcal{L}_V f_i) \wedge df_{i+1} \wedge ... \wedge df_{n-2} = 0.$$

To proceed we could rely on Liouville's Theorem [11], but we choose to conclude in a self-contained way using just basic differential properties. To simplify the notation, we call $\Omega = \alpha \wedge dx^{n-1} \wedge dx^n$. Let's compute

$$\begin{aligned} 0 = \mathcal{L}_V \mu = \mathcal{L}_V(\rho\Omega) = \mathcal{L}_V(\rho)\Omega + \rho\alpha \wedge (\mathcal{L}_V(dx^{n-1} \wedge dx^n)) = \\ = \Big[\Big(\frac{\partial \rho}{\partial x_{n-1}} V^{n-1} + \frac{\partial \rho}{\partial x_n} V^n\Big)\Omega + \rho\alpha \wedge \Big(dV^{n-1} \wedge dx^n + dx^{n-1} \wedge dV^n\Big)\Big] = \\ = \Big[\frac{\partial(\rho V^{n-1})}{\partial x_{n-1}} + \frac{\partial(\rho V^n)}{\partial x_n}\Big]\Omega. \end{aligned}$$

This implies that the term multiplying Ω in the last line vanishes at any point $(x_1, ..., x_n) \in M$. Repeating the same computations with

$$x_1 = c_1, ..., x_{n-1} = c_{n-2}$$

fixed, we get that even $\mathcal{L}_X \omega = 0$.

$\square$

At this point we have a volume form ω on N which is invariant for the vector field $X = V|_N$. We can look at such a 2−form as a symplectic structure on this invariant set. We now introduce on N a multi-valued Hamiltonian function H.

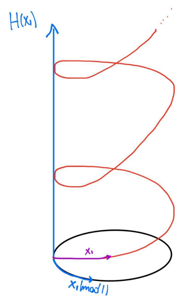


Figure 4.1: Situation where $H : \mathbb{R} \to \mathbb{R}$ is not 1−periodic

To be clear, let's specify that by multi-valued function we mean that H is not 1−periodic in its variables when seen as a function from $\mathbb{R}^2$ to $\mathbb{R}$. This is the same situation which happens for functions defined on $\mathcal{S}^1$ which are not $1-$periodic and can be described as in Figure 4.1. Hence locally on N there is a classical Hamiltonian function and we have $dH = i_X\omega$ (precisely, this function exists since we are working on the covering space $\mathbb{R}^2$ of the torus which is simply connected, but when thought as a function on the torus it is multi-valued). On the other hand, the $1-$form dH is single valued and, if γ_1 is the cycle $\{(x_{n-1}, 0) : x_{n-1} \in \mathcal{S}^1\}$ of N and similarly γ_2 corresponds to the angular coordinate x_n, it reads

$$\begin{aligned} dH &= \Big(\int_{\gamma_1} dH\Big) dx^{n-1} + \Big(\int_{\gamma_2} dH\Big) dx^n + dh(x_{n-1}, x_n) \\ &= c_1 dx^{n-1} + c_2 dx^n + dh(x_{n-1}, x_n). \end{aligned}$$

Here (x_{n-1}, x_n) are $1-$periodic "angular" coordinates on N, while h is $1-$periodic in both its entries when seen as a function from $\mathbb{R}^2$ to $\mathbb{R}$. We can then recover the multi-valued function H as follows:

$$H = c_1 x_{n-1} + c_2 x_n + h(x_{n-1}, x_n).$$

At this point the aim is to find two linearly independent commuting vector fields on N so that we can prove that

- N is diffeomorphic to $\mathbb{T}^2$,
- there are angular coordinates with respect to which the dynamics is a linear flow,
- we can extend this construction on the tori in some tubular neighbourhood of N.

First of all let's introduce the Riemannian metric $g = (dx^{n-1})^2 + (dx^n)^2$ on N and define the vector field

$$\xi = \frac{1}{|\nabla H|} \nabla H.$$

We notice that this vector field, which is globally defined on N, is pointwise orthogonal to X with respect to the metric g. Indeed,

$$i_X\omega = \tilde{\rho}\Big[dx^{n-1}(X)dx^n - dx^n(X)dx^{n-1}\Big] = c_1 dx^{n-1} + c_2 dx^n + dh,$$

which implies

$$\begin{aligned} g(X,\xi) &= dx^{n-1}(X)dx^{n-1}(\xi) + dx^n(X)dx^n(\xi) = \\ &= \frac{1}{\tilde{\rho}|\nabla H|}\Big[\Big(c_2 + \frac{\partial h}{\partial x_n}\Big)\Big(c_1 + \frac{\partial h}{\partial x_{n-1}}\Big) - \Big(c_1 + \frac{\partial h}{\partial x_{n-1}}\Big)\Big(c_2 + \frac{\partial h}{\partial x_n}\Big)\Big] = 0 \end{aligned}$$

where everything is evaluated at a point $n \in N$. This implies that the flow of the vector field ξ preserves the trajectories of X. To show this fact let's notice that since $\mathcal{L}_X H = 0$, then the level sets $\{H = h\}$ are invariant with respect to the flow of X. But since N is $2-$dimensional, the trajectories of X coincide with these level sets[1]. So flowing along an integral curve of ξ for a certain time

[1] In case these level sets are closed curves, since they do not contain any equilibrium point, they coincide with closed trajectories. If they are open subsets, then they are connected spaces and a trajectory is both open and closed and hence it coincides with the whole level set.

$t = c$ we pass from the trajectory $\gamma_h = \{H = h\}$ to $\gamma_{h+c} = \{H = h + c\}$. From this follows that a trajectory of X is closed if and only if all the others are closed.

Let's notice that we still do not have a dynamical symmetry for X. Consider now the vector field

$$\tilde{\xi} = \xi + \lambda X.$$

Proposition 24. *There exists a $\lambda \in \mathbb{R}$ such that the vector field $\tilde{\xi}$ has at least a closed trajectory α which is non-homotopically trivial and transversal to the vector field X.*

Proof. To prove this result we first of all need to prove that there is a closed transversal α for the nowhere vanishing vector field X.

Recall that homotopies are homeomorphisms so since X has no orbits of the type $Orb(x_0) = \{x_0\}$, then there are no homotopically trivial closed trajectories since they could be continuously deformed into the trivial loop.

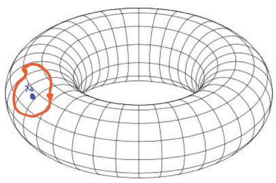
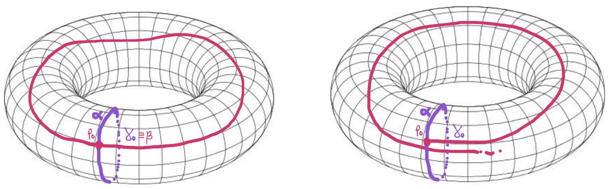


Figure 4.2: Not admissible case

Let's now consider what happens in the neighbourhood of a point $P_0 \in N$ lying on the trajectory $\gamma_0 = \{H = 0\}$. The possibilities are only two as presented in Figure (4.3).



(a) γ_0 closed (b) γ_0 not closed

Figure 4.3: Two possibilities to build α.

1. γ_0 is not closed. Then we can take as α an arbitrary non-trivial cycle on N transversal to X.

2. γ_0 is closed and defines itself a non-homotopically trivial closed loop β. Then we can define α as an arbitrary non-trivial loop complementary to β.

We now lift the trajectory $\gamma_0 = \{H = 0\}$ passing through the point $P_0 \in N$ at time $t = 0$ on the covering space $\mathbb{R}^2$ of the torus N. In this covering space the closed loop α defines a translation, i.e. it is a vector connecting the point

$P_0 \in [0,1) \times [0,1)$ to another point P_0' in the lattice $\pi^{-1}(P_0) \simeq \mathbb{Z}^2$ where $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 \simeq N$ is the covering map of N.

Now the proof of this Proposition is completed because of the two following facts:

- Suppose $H(P_0') = c$ and hence $P_0' \in \gamma_c$. We can pass from γ_0 to γ_c flowing for a time $t = c$ along the integral curve of ξ passing through P_0 at time $t = 0$. So we move to $P' = \Phi_c^{\xi}(P_0) \in \gamma_c$.
- We can now move from P' to P_0' flowing along the trajectory γ_c for a certain time $\bar{\lambda}$. So we fix it in such a way that $\Phi_{\bar{\lambda}}^{X}(P') = P_0'$. This implies $\Phi_1^{c\xi+\bar{\lambda}X}(P_0) = P_0'$.

Since we know that on the $2-$torus to transition from P_0 to P_0' means flowing along the closed trajectory α, we get that the vector field $\tilde{\xi}$ satisfies the Proposition. □

We now consider this loop α as a Poincaré section for the transversal vector field X. Recall the two following facts:

- The trajectory γ_0 of X is closed if and only if all the others are,
- If a fixed-point-free flow with a non periodic recurrent orbit on a closed surface admits a closed transversal, then every orbit intersects the transversal. (Proof in Proposition 14.2.2 at page 458 of [1]).

This means there is a well defined first return map $r : \mathcal{S}^1 \to \mathcal{S}^1$ (where $\alpha \simeq \mathcal{S}^1$). Introduce an angular coordinate $h \in [0,c)$ along the closed loop α and define the function $T : [0,c) \to \mathbb{R}^+$ which maps each trajectory γ_h of X into the time $T(h)$ required to come back to α. We now reparametrize the time of the flow along the trajectories of X in such a way that for any $h \in [0,c)$ the first return map is always 1.

Let's introduce the new time coordinate as follows:

$$t(\tau) = \int_0^{\tau} (1 + s(h)\psi(x))dx,$$

where

$$s(h) = \frac{T(h) - 1}{\int_0^1 \psi(x)dx}.$$

This choice implies $t(1) = T(h)$, i.e. if in the original time coordinate t we needed the time $T(h)$ to come back to α moving along γ_h, now we just need the time $\tau = 1$ (completely independent on h and hence on the chosen trajectory).

$\psi(t)$ is a classical regularization function and it is necessary to obtain a smooth change of variable on N. Let's show the plot of this regularization function and then check why this specific choice allows $t = t(\tau)$ to be a well defined change of coordinates. In Figure 4.4 there is an example of the kind of function we need.

This is a well defined and smooth change of coordinates on χ because of two reasons:

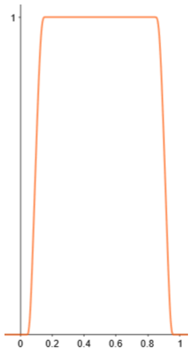


Figure 4.4: Possible choice of smoothing function, built with Geogebra as follows: `https://www.geogebra.org/m/b489nyuw`

1. It is invertible since we can make $\int_0^1 \psi(t)dt$ as near as we want to 1. This guarantees $s(h) > -1$ and hence
$$t'(\tau) = 1 + s(h)\psi(\tau) > 0.$$
From this follows the invertibility of the transformation.

2. It is smooth on N.
$$\frac{\partial \tau}{\partial t}(t, h) = \frac{1}{1 + s(h)\psi(\tau)},$$
so in a neighborhood of $\tau = 0$ and $\tau = 1$, having $\psi = 0$, the first derivative is continuous and so are higher order derivatives.

Remains to check it is possible to approximate as well as desired this change of coordinates with a real-analytic function (supposing to have real-analyticity of the other functions involved in the system). This result is based on one of the Whitney extension theorems (see [6]).

Theorem 12. *Let $f : M \to \mathbb{R}$ be a smooth function on a real-analytic compact manifold M. Then there exists a real-analytic function $g : M \to \mathbb{R}$ that approximates f together with all its derivatives up to a certain order k.*

Hence the change of coordinates on the manifold N can be supposed to be real-analytic. At this point we have a well defined pair of angular coordinates τ and h which allow to coordinatize the torus. The important thing is that the trajectories of $\partial/\partial\tau$ coincide with those of X, but they are covered in different times so their orbits do not coincide. Namely, $\partial/\partial\tau$ is a time reparametrization of X. Therefore, there exists a positive scalar function $\Phi(\tau, h)$ such that
$$\partial_\tau = \Phi X.$$
We get two linearly independent commuting vector fields:
$$[\partial_h, \Phi X] = [\partial_h, \partial_\tau] = \frac{\partial^2}{\partial h \partial \tau} - \frac{\partial^2}{\partial h \partial \tau} = 0.$$
This gives that N is diffeomorphic to $\mathbb{T}^2$ since it is a smooth $2-$manifold admitting two tangent vector fields which are independent and commute. Proceeding

exactly as in the local proof of Bogoyavlensky's Theorem, we get that here the field ΦX writes

$$\Phi X = \lambda_1 \frac{\partial}{\partial \alpha_1} + \lambda_2 \frac{\partial}{\partial \alpha_2}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $(\alpha_1, \alpha_2) \in \mathbb{T}^2$ angular coordinates. This allows to conclude:

$$\dot{\alpha}_1 = \frac{\lambda_1}{\Phi(\alpha_1, \alpha_2)}, \quad \dot{\alpha}_2 = \frac{\lambda_2}{\Phi(\alpha_1, \alpha_2)}.$$

All the constructions done locally on the invariant level set are differentiable. This means that we can extend in a differentiable way the commuting fields from an invariant torus to those in a tubular neighbourhood of N. On each invariant torus the symmetries are:

$$\begin{aligned} \frac{\partial}{\partial \tau} &= \Phi X, \\ \frac{\partial}{\partial h} &= \tilde{\xi} = \frac{1}{|\nabla H|} H + \bar{\lambda} X. \end{aligned}$$

We can consider an open neighbourhood $B = \mathcal{B}^{n-2}(\boldsymbol{c})$ of $\boldsymbol{c} \in \mathbb{R}^{n-2}$ and extend them on $F^{-1}(B) = \mathcal{U}(N) \simeq \mathbb{T}^2 \times B$ (recalling that by Ehresmann's Theorem F is a locally-trivial fibration) since they both preserve the first integrals of X. Considering

$$\theta : \mathbb{R}^2 \times \mathcal{U}(N) \to \mathcal{U}(N),$$

the extension of the action studied on N to $\mathcal{U}(N)$, we can proceed as in the semi-global extension of Bogoyavlensky.

$\square$

Chapter 5

Comparison between Bogoyavlensky and Euler-Jacobi

In this Chapter we are going to study the relations between the two integrability Theorems presented in Chapters 3 and 4. This comparison is motivated by the similarities in the assumptions and the outcomes characterizing the two results. This Chapter is mainly devoted to the analysis of a single question: is there some sufficient condition ensuring that a system integrable in the sense of Euler-Jacobi is even B-integrable? Before focusing on this question, we briefly study when a system integrable in the sense of Bogoyavlensky is integrable in the sense of Euler-Jacobi, which is an easier problem.

5.1 From Bogoyavlensky to Euler-Jacobi

In this short section we highlight a situation in which broad-integrability implies the invariance of a volume form, and hence Euler-Jacobi integrability. Indeed, any $(2, n-2)$-integrable system defined on a smooth, orientable manifold of dimension n is also integrable à la Euler-Jacobi. Consider the system of coordinates

$$x_1 = f_1, ..., x_{n-2} = f_{n-2}, x_{n-1}, x_n,$$

with respect to which the two symmetry fields of the system write

$$X = X^1 \frac{\partial}{\partial x_{n-1}} + X^2 \frac{\partial}{\partial x_n},$$
$$Y = Y^1 \frac{\partial}{\partial x_{n-1}} + Y^2 \frac{\partial}{\partial x_n}.$$

$$X \wedge Y = (X^1 Y^2 - X^2 Y^1) \frac{\partial}{\partial x_{n-1}} \wedge \frac{\partial}{\partial x_n},$$

where $X^1 Y^2 - X^2 Y^1 \neq 0$ since $X \wedge Y \neq 0$ by assumption. To proceed, we need to make an important remark. There is a $1-1$ relation between non-degenerate

2−vector fields and 2-forms. To be precise,

$$(dx \wedge dy)(X_f, X_g) = \Big(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\Big)(df, dg)$$

where $i_{X_f}(dx \wedge dy) = -df$ (and the same for g). Thanks to this identification, we can interpret the wedge product of the two vector fields X and Y as a differential 2-form. This allows us to consider the following volume form

$$\begin{aligned}\mu &= df_1 \wedge ... \wedge df_{n-2} \wedge X \wedge Y = \\ &= (X^1Y^2 - X^2Y^1) df_1 \wedge ... \wedge df_{n-2} \wedge dx^{n-1} \wedge dx^n\end{aligned}$$

and show it is conserved by X:

$$\begin{aligned}\mathcal{L}_X \mu = \Big(\sum_{i=1}^{n-2} d(X(f_i)) df_1 \wedge ... \wedge df_{i-1} \wedge df_{i+1} \wedge ... \wedge df_{n-2}\Big) \wedge X \wedge Y + ... \\ ... + df_1 \wedge ... \wedge df_{n-2} \wedge ([X, X] \wedge Y + X \wedge [X, Y]) = 0.\end{aligned}$$

From this follows that the system is also integrable in the sense of Euler-Jacobi.

Let's conclude showing that the field

$$X = p_x \partial_x + p_y \partial_y - \frac{y p_x p_y}{1 + y^2} \partial_{p_x} - y \partial_{p_y}$$

presented in Section 3.3 is integrable in the sense of Euler-Jacobi by explicitly finding an invariant measure. Indeed, we already know that this system is integrable in the sense of Euler-Jacobi, but the construction developed below is still interesting because we recover the explicit expression of an invariant volume form for the vector field. This system is defined on the 4−dimensional manifold $M = \mathcal{S}^1 \times \mathbb{R} \times \mathbb{R}^2$. It admits the two first integrals E and J presented previously. To highlight the existence of a smooth invariant volume form, let's perform the following change of coordinates

$$(x, y, p_x, p_y) \to (x, y, m_x, m_y)$$

where:

$$m_x = (1 + y^2) p_x, \quad m_y = p_y.$$

So $p_x = m_x/(1 + y^2)$ and the first equation becomes

$$\dot{x} = \frac{m_x}{1 + y^2}.$$

The second is simply $\dot{y} = m_y$. Then

$$\dot{p}_x = \frac{\dot{m}_x}{1 + y^2} - 2y m_x \frac{\dot{y}}{(1 + y^2)^2} = -\frac{y m_x m_y}{(1 + y^2)^2}$$

implies $\dot{m}_x(1 + y^2) - 2y m_x m_y = -y m_x m_y$, i.e.

$$\dot{m}_x = \frac{y m_x m_y}{1 + y^2}.$$

Moreover, $\dot{m}_y = \dot{p}_y = -y$. This means the vector field in the new coordinates writes

$$\tilde{X} = \frac{m_x}{1+y^2}\partial_x + m_y\partial_y + \frac{y m_x m_y}{1+y^2}\partial_{m_x} - y\partial_{m_x}.$$

Let's now check that this vector field admits the following invariant volume form:

$$\mu = \frac{1}{\sqrt{1+y^2}} dx \wedge dy \wedge dm_x \wedge dm_y = f(y) dx \wedge dy \wedge dm_x \wedge dm_y.$$

By Cartan's magic formula and $d\mu = 0$ we get $\mathcal{L}_{\tilde{X}}\mu = d(i_{\tilde{X}}\mu)$.

$$\begin{aligned} i_{\tilde{X}}\mu &= f(y)\Big[i_{\tilde{X}}(dx \wedge dy) \wedge dm_x \wedge dm_y + dx \wedge dy \wedge i_{\tilde{X}}(dm_x \wedge dm_y)\Big] = \\ &= f(y)\Big[\frac{m_x}{1+y^2} dy \wedge dm_x \wedge dm_y - m_y dx \wedge dm_x \wedge dm_y + ... \\ &.. + \frac{y m_x m_y}{1+y^2} dx \wedge dy \wedge dm_y - y dx \wedge dy \wedge dm_x\Big]. \end{aligned}$$

So we get

$$\begin{aligned} d(i_{\tilde{X}}\mu) &= \Big[m_y f'(y) + f(y)\frac{y m_y}{1+y^2}\Big] dx \wedge dy \wedge dm_x \wedge dm_y \\ &= \Big[-\frac{y m_y}{(1+y^2)^{3/2}} + \frac{y m_y}{(1+y^2)^{3/2}}\Big] dx \wedge dy \wedge dm_x \wedge dm_y = 0. \end{aligned}$$

This means that since first integrals are preserved by coordinate changes, all the hypotheses of Euler-Jacobi integrability Theorem hold. Hence the vector field X is integrable in the sense of Euler-Jacobi.

5.2 From Euler-Jacobi to Bogoyavlensky

In this Section, we recover some sufficient conditions ensuring when a system integrable in the sense of Euler-Jacobi is broadly-integrable too. Given a $n-$dimensional smooth and orientable manifold M and a smooth vector field $X \in \mathfrak{X}(M)$ with $(n-2)$ functionally independent first integrals, is there some relation between the integrability of X with respect to the two approaches? The reason why this question is quite natural, is that both the Theorems above guarantee the quasi-periodicity of the flow on the invariant tori defined by the fibration via the first integrals, but it may happen that a time reparametrization of the vector field is required. Having an invariant volume form for the system, allows to get the hypotheses of Euler-Jacobi's Theorem. In this case integrability comes at the cost of reparametrizing in time. On the other hand, the presence of an additional "interesting" symmetry field Y, allows quasi-periodicity with no required rescaling of the field X. Here, by interesting symmetry field Y, we mean a field $Y \in \mathfrak{X}(M)$ such that:

- $X \wedge Y \neq 0$ on M,
- $[X, Y] = 0$,
- $L_Y f_i = 0$ for all the first integrals of X.

The question is if there is some condition we can add to systems integrable in the sense of Euler-Jacobi to be sure they are even B-integrable. We now consider the case where $M = \mathbb{T}^2$. In this case in principle no first integral is required to proceed in our analysis, since the dimension of M is $n = 2$ and $n - 2 = 0$. Let now $X \in \mathfrak{X}(M)$ be a smooth vector field which can be expressed in the angular coordinates $(x, y) \in \mathcal{S}^1 \times \mathcal{S}^1$ as follows:

$$X = X^1(x, y)\partial_x + X^2(x, y)\partial_y.$$

If we see the two components X^i of X as functions from $\mathbb{R}^2$ to $\mathbb{R}$, then they must be $1-$periodic in both their entries. Suppose X admits an invariant volume form $\omega = p(x, y)dx \wedge dy$, with $p(x, y)$ $1-$periodic in both its entries. Hence X is integrable in the sense of Euler-Jacobi.

Let's now study when the conservation of this volume form allows to construct a second symmetry field of X. There are vector fields where the independent symmetry does not need any particular work to be found. Let's define the generic field $X \in \mathfrak{X}(\mathbb{T}^2)$:

$$X = X^1(x, y)\partial_x + X^2(x, y)\partial_y$$

and list below some of the evident cases:

- If $X^i(x, y) = f_i(x)$, with $f_2(x) \not\equiv 0$, then $Y = \partial_y$ is an independent symmetry for X.
- If $X^i(x, y) = g_i(y)$, with $g_1(y) \not\equiv 0$, then $Y = \partial_x$ is an independent symmetry for X.

This has an important consequence in terms of integrability, indeed these fields are B-integrable. Apart from these simple cases, we now analyze more general vector fields. We will consider various approaches starting with some assumption on the density of the invariant measure. First of all we take into account the case in which the density function is $p(x, y) = c \in \mathbb{R}$. When the field X satisfies $L_X\omega = 0$ with this choice of the density function, we can immediately recover a particular structure for the components of the field. X must be as follows

$$X = f(y)\frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y},$$

where $f(y) = f(y + 1)$ and $g(x) = g(x + 1)$ for any $(x, y) \in \mathbb{R}$. This constraint is mainly due to the $1-$periodicity condition. We can check it preserves the $2-$form $\omega = cdx \wedge dy$ for any $c \in \mathbb{R}$ since

$$\mathcal{L}_X\omega = d(i_X\omega) = cd(f(y)dy - g(x)dx) = 0.$$

To show that if X preserves ω, then it has this structure, we proceed by direct computation as follows:

$$\mathcal{L}_X\omega = c[d(X^1(x, y)dy - X^2(x, y)dx)] = c\Big(\frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y}\Big)dx \wedge dy = 0, \quad (5.1)$$

which brings to

$$X^1(x, y) = h(y) - \int_0^x \frac{\partial X^2(s, y)}{\partial y}ds$$

for an arbitrary $1-$periodic function $h : \mathbb{R} \to \mathbb{R}$. But X^1 must be $1-$periodic in x, so we need to impose $X^1(x+1, y) = X^1(x, y)$.

$$X^1(x+1, y) = X^1(x, y) + \int_0^1 \frac{\partial X^2(s, y)}{\partial y} ds$$

implies the condition $X^2(x, y) = g(x)$ for some $g : \mathbb{R} \to \mathbb{R}$ which is $1-$periodic. Then from Equation (5.1) follows $X^1(x, y) = f(y)$ for another function f again $1-$periodic. A vector field of this kind is integrable in the sense of Euler-Jacobi, so up to a time reparametrization it takes the form

$$\dot{u} = \frac{\lambda}{\Phi(u, v)}, \quad \dot{v} = \frac{\mu}{\Phi(u, v)},$$

with Φ a positive $1-$periodic function. The question is: is it even B-integrable? The answer would be yes if we were able to define a second vector field Y which is independent from X, such that $[X, Y] = 0$. The answer is no in general. Let's consider this vector field:

$$X = [\sin(2\pi y) + 2]\partial_x. \tag{5.2}$$

Proposition 25. *All the vector fields commuting with X depend linearly on it.*

Proof. Let $Y = f(x, y)\partial_x + g(x, y)\partial_y$ be an arbitrary vector field on $M = \mathbb{T}^2$, hence

$$f(x+1, y+1) = f(x, y) \text{ and } g(x+1, y+1) = g(x, y).$$

Imposing the commutativity property we get

$$\begin{aligned} 0 = [X, Y] &= [\sin(2\pi y) + 2]\frac{\partial f}{\partial x}\partial_x + [\sin(2\pi y) + 2]\frac{\partial g}{\partial x}\partial_y - g\frac{\partial(\sin(2\pi y))}{\partial y}\partial_x = \\ &= [\sin(2\pi y) + 2]\frac{\partial g}{\partial x}\partial_y + \Big([\sin(2\pi y) + 2]\frac{\partial f}{\partial x} - 2\pi g\cos(2\pi y)\Big)\partial_x. \end{aligned}$$

This implies that

$$[\sin(2\pi y) + 2]\frac{\partial g}{\partial x} = 0,$$

and hence necessarily $g(x, y) = g(y)$. Since even the first component has to vanish we get

$$\frac{\partial f}{\partial x} = 2\pi g(y)\Big(\frac{\cos(2\pi y)}{2 + \sin(2\pi y)}\Big),$$

namely

$$f(x, y) = 2\pi g(y)\Big(\frac{\cos(2\pi y)}{2 + \sin(2\pi y)}\Big)x + h(y)$$

which is $1-$periodic in the x variable if and only if $g \equiv 0$ in fact

$$\begin{aligned} f(x+1, y) - f(x, y) = 2\pi g(y)\Big(\frac{\cos(2\pi y)}{2 + \sin(2\pi y)}\Big)(x+1) + h(y) - ... \\ + 2\pi g(y)\Big(\frac{\cos(2\pi y)}{2 + \sin(2\pi y)}\Big)x - h(y) = \\ = 2\pi g(y)\Big(\frac{\cos(2\pi y)}{2 + \sin(2\pi y)}\Big) = 0 \end{aligned}$$

if and only if $g \equiv 0$. $\square$

This field preserves the measure $\omega = cdx \wedge dy$ ($c \in \mathbb{R}$) and those with a density function just depending on y, being hence integrable in the sense of Euler-Jacobi. Since we can not build a second symmetry for the system, up to looking for first integrals of the vector field, it is not integrable in the sense of Bogoyavlensky. Actually, this vector field is B-integrable since it admits the first integral $f(x, y) = \sin(2\pi y)$ satisfying the assumptions of Theorem 8, but this example is interesting because it suggests that not all the vector fields have symmetries and hence that in general there is not an immediate relation between Euler-Jacobi integrability and B-integrability.

Moreover, considering $\tilde{X} = [\sin(2\pi x) + 2]\partial_y \in \mathfrak{X}(M)$, we see that even the conservation of a volume form with density just depending on x is not enough for an Euler-Jacobi integrable system to be B-integrable.

This suggests that imposing some condition on the invariant measure may not be the best approach to the problem. Clearly, the example above does not give any information about measures with densities of the type $p(x, y) = a(x) + b(y)$ or $p(x, y) = a(x)b(y)$ or in general depending on both the angular coordinates. However, in these cases, computations are quite complicated and due to the periodicity constraint we usually encounter trivial symmetries.

Up to now we have seen that a vector field preserving an arbitrary smooth measure with density depending at most on one of the two angular coordinates, is not sufficient to guarantee the existence of a second symmetry for the system and hence its B-integrability.

Another immediate consequence of the previous computations is that even when the invariant volume form ω is exact, i.e. $d\alpha = \omega$ for some $\alpha \in \Lambda^1(M)$, the field X is not B-integrable in general or, more precisely, in general it does not have a linearly independent dynamical symmetry. Indeed, we can set up the following example:

$$\begin{aligned} &\alpha = -f(y)dx, \quad \text{with } f(y) = f(y+1), \\ &\omega = d\alpha = f'(y)dx \wedge dy \end{aligned}$$

so that we have a density just depending on y and hence ω is preserved by our field X defined before in (5.2). This field does not have relevant symmetries, so exactness is not a sufficient condition giving the missing link between integrability à la Euler-Jacobi and broad-integrability.

Instead of looking at particularly designed density functions $p = p(x, y)$, we now change our approach to the problem. In the remaining part of this Chapter we will introduce three conditions without specifying the density function. The former involves the presence of a first integral of the dynamics, while the other two are fully based on constructing a linearly independent dynamical symmetry of the vector field defining the dynamics. Precisely, the second and third conditions give an answer to the question: is the time reparametrization in Euler-Jacobi Theorem always required?

5.2.1 First density independent condition

Suppose there exists a smooth function $m : \mathbb{T}^2 \to \mathbb{R}$ such that

$$\mathcal{L}_X m = k \in \mathbb{R}.$$

Consider now a vector field $Y \in \mathfrak{X}(M)$ such that

$$i_Y \omega = dm.$$

Then Y is a symmetry for X. This means that if for example the vector field X does admit a first integral $m \in \mathcal{C}^\infty(M)$, then $L_X m = 0$ and hence if we set Y as above we get

$$i_{[X,Y]}\omega = L_X(i_Y\omega) - i_Y(L_X\omega) = L_X(dm) = d(L_X m) = 0.$$

This implies that necessarily m gives rise to a symmetry field of X, which is not in general guaranteed to be independent from X. Let's explicitly construct this symmetry field and then conclude with a couple of examples. Suppose $m = m(x, y)$ is a first integral of X and $\omega = p(x, y)dx \wedge dy$. Then we impose

$$i_Y\omega = p(x, y)\Big(Y^1(x, y)dy - Y^2(x, y)dx\Big) = \frac{\partial m}{\partial x}(x, y)dx + \frac{\partial m}{\partial y}(x, y)dy,$$

and hence we set

$$Y = \frac{1}{p(x, y)}\Big[\frac{\partial m}{\partial y}\partial_x - \frac{\partial m}{\partial x}\partial_y\Big],$$

which is a reasonable construction since the density function never vanishes. The fact that $X \wedge Y$ may or may not be zero, strictly depends on the specific considered dynamical system. Moreover, the condition

$$L_X m = k \in \mathbb{R}$$

does not always degenerate to the case $k = 0$. This depends on the manifold where we are working. Suppose for a moment the dynamics is defined on $\mathbb{R}^2$, and consider:

$$X = xy\partial_x + \partial_y \in \mathbb{R}^2.$$

The function

$$m(x, y) = cy$$

gives

$$\mathcal{L}_X m = c \in \mathbb{R},$$

even if in general it is not a first integral of X. The problem for fields defined on the $2-$torus, is again the periodicity constraint. The example built above works just because we have considered an affine function m. On the other hand, a function $m : \mathbb{T}^2 \to \mathbb{R}$ with $dm = adx + bdy$, $a, b \in \mathbb{R}$, implies necessarily $m \equiv 0$. This is why $L_X m = k$, even if it seems a milder constraint than the one with $k = 0$, does not give rise to other classes of B-integrable systems on the $2-$torus.

Before moving to the examples, it is important to remark that even if the first integral m may give rise to a linearly dependent dynamical symmetry, when such a m does exist and its level sets satisfy the assumption of Theorem 8, the vector field is B-integrable. Precisely, since the vector field X is a symmetry for itself and it admits a first integral, then X is integrable of type $(1, 1)$. The only problem is that this procedure may not give rise to an independent symmetry. By the way, since the focus of the analysis we are doing is on the presence of relevant dynamical symmetries, the following examples show the limits of this approach in building them.

The construction of first integrals of a vector field X is not a trivial operation. Some strategies to find them for measure preserving fields (i.e. fields having an integrating factor) are presented in [8]. We now consider a Hamiltonian vector field X with respect to the $2-$form ω. Let

$$H = \sin(2\pi x) + \cos(2\pi y),$$

so $X_H = f(x, y)\partial_x + g(x, y)\partial_y$ is such that

$$i_{X_H}[p(x, y)dx \wedge dy] = -dH = -2\pi[\cos(2\pi x)dx - \sin(2\pi y)dy].$$

This implies

$$p(x, y)[f(x, y)dy - g(x, y)dx] = -2\pi[\cos(2\pi x)dx - \sin(2\pi y)dy]$$

and hence

$$X_H = \frac{2\pi}{p(x, y)}\Big[\sin(2\pi y)\partial_x + \cos(2\pi x)\partial_y\Big].$$

This vector field preserves $\omega = p(x, y)dx \wedge dy$ and has even a first integral given by the function H. The symmetry field Y we can build following the previous procedure is the one satisfying

$$i_Y\omega = dH.$$

We immediately see that $[X, Y] = 0$ but $Y \wedge X = 0$ too. Another example which highlights the limits of this approach is based on the same vector field studied before:

$$X = [\sin(2\pi y) + 2]\partial_x.$$

Indeed, X admits the first integral $m(x, y) = \sin(2\pi y)$ but since X does not have linearly dependent symmetries, necessarily m gives rise to non relevant symmetries. On the other hand, this does not say that X is not B-integrable, precisely we can say that it is not $(2, 0)-$integrable but it is $(1, 1)-$integrable.

5.2.2 Second density independent condition

Let $X \in \mathfrak{X}(\mathbb{T}^2)$ be a smooth vector field preserving the $2-$form $\omega \in \Lambda^2(M)$ with $M = \mathbb{T}^2$. It admits a symmetry field if and only if it preserves even a $1-$form $\alpha \in \Lambda^1(M)$, i.e. $\mathcal{L}_X\alpha = 0$. Precisely, setting $Y \in \mathfrak{X}(M)$ in such a way that

$$i_Y\omega = \alpha,$$

guarantees that $[X, Y] = 0$. This is not in general a sufficient condition for the B-integrability of X, which now just depends on the linear independence of X

from Y. Before going on, let's recall why we are sure to have a symmetry field for X:

$$i_{[X,Y]}\omega = \mathcal{L}_X(i_Y\omega) - i_Y(\mathcal{L}_X\omega) = \mathcal{L}_X(\alpha) = 0.$$

Since the term $\mathcal{L}_X\omega$ vanishes when ω is an invariant measure for X, each and every symmetry field of X needs to be such that $\mathcal{L}_X(i_Y\omega) = 0$. The problem is that we may have that, as for the example written above $X = [\sin(2\pi y) + 2]\partial_x$, there is no independent symmetry field. This means that this relation, even if it is satisfied by some vector field Y, does not necessarily give some interesting information about the dynamics.

This suggests that the presence of an invariant $1-$form for an Euler-Jacobi integrable system is almost the missing piece connecting Euler-Jacobi integrability with broad-integrability. Let's specify that the expression describing Y in this case, supposing $\alpha = a(x,y)dx + b(x,y)dy$ the conserved $1-$form, is the following one

$$Y = \frac{b(x,y)}{p(x,y)}\partial_x - \frac{a(x,y)}{p(x,y)}\partial_y.$$

The condition $X \wedge Y = 0$ is equivalent to the existence of a function $f \in \mathcal{C}^\infty(\mathbb{T}^2)$ with $Y = fX$. Since f stands for a time reparametrization of X, we can even suppose $f > 0$. Assume $Y = fX$, then

$$\alpha = i_Y\omega = i_{fX}\omega = f i_X\omega.$$

Calling $g = 1/f$, we get $i_X\omega = g\alpha$. This result implies the following one

$$d(i_X\omega) = \mathcal{L}_X\omega = 0 = d(g\alpha) = dg \wedge \alpha + gd\alpha = dg \wedge \alpha \tag{5.3}$$

when $d\alpha = 0$. Since setting $g \equiv const$ gives a solution to Equation (5.3), when α is closed we are not sure to find a field Y which does not depend on X.

Proposition 26. *Consider a smooth vector field $X \in \mathfrak{X}(\mathbb{T}^2)$ satisfying the following properties:*

1. $\mathcal{L}_X\omega = 0$ *for a smooth measure* $\omega \in \Lambda^2(\mathbb{T}^2)$*,*
2. $\mathcal{L}_X\alpha = 0$ *for some* $\alpha \in \Lambda^1(\mathbb{T}^2)$ *satisfying one of the two following properties:*
   - *it is not proportional to $\beta = i_X\omega$, i.e. there is no $f \in \mathcal{C}^\infty(\mathbb{T}^2)$ such that $f\beta = \alpha$ or*
   - *it vanishes at a point $p \in \mathbb{T}^2$ where $(d\alpha)_p \neq 0$.*

Then the field $Y \in \mathfrak{X}(M)$ such that $i_Y\omega = \alpha$, satisfies

1. $X \wedge Y \neq 0$*,*
2. $[X,Y] = 0$*,*

and hence X is not just integrable in the sense of Euler-Jacobi, but even integrable in the sense of Bogoyavlensky.

Proof. First of all let's show that $[X, Y] = 0$. This follows immediately from this computation:

$$i_{[X,Y]}\omega = i_{\mathcal{L}_X Y}\omega = \mathcal{L}_X(i_Y\omega) - i_Y(\mathcal{L}_X\omega) = \mathcal{L}_X\alpha = 0.$$

Since ω is non-degenerate, necessarily this implies $[X, Y] = 0$.

By contradiction let's assume there exists a function $f : \mathbb{T}^2 \to \mathbb{R}$ such that $Y = fX$, which by construction must be always different from 0 (being a time reparametrization). This implies

$$\alpha = i_Y\omega = i_{fX}\omega = f i_X\omega.$$

Calling $\beta = i_X\omega$, we have that X is proportional to Y if and only if α is proportional to β. Hence if the form β does not read $f\alpha = \beta$ for any f, then the symmetry Y we find is a relevant one as stated in the proposition.

To verify the other condition for α, let's set $g = 1/f$. This means $g\alpha = i_X\omega$. When this is true, necessarily follows:

$$di_X\omega = \mathcal{L}_X\omega = 0 = d(g \wedge \alpha) = dg \wedge \alpha + gd\alpha.$$

If there exists a point $p \in \mathbb{T}^2$ such that $\alpha|_p = 0$, then this equation at p reads:

$$g(d\alpha)|_p = 0.$$

Having $(d\alpha)_p \neq 0$, necessarily $g(p) = 0$. This is not possible in our case since $g(p) = 1/f(p)$ can not be 0. Hence this implies there is no global solution $f \in \mathcal{C}^\infty(\mathbb{T}^2)$. We conclude the symmetry Y is independent from X, which is hence B-integrable. $\square$

Let's now check that in one of the cases where the symmetry is evident (those listed above), one of the two conditions in the Theorem holds. We can consider for example a field of the kind

$$X = f(y)\partial_x + \lambda\partial_y,\ \lambda \neq 0,$$

which admits the symmetry field $Y = \partial_x$. This field preserves the volume form $\omega = dx \wedge dy$ and the form

$$\alpha = i_Y\omega = [Y^1 dy - Y^2 dx] = -dx.$$

Moreover, $\beta = i_X\omega = [X^1 dy - X^2 dx] = [f(y)dy - \lambda dx]$. Hence in this case there is no rescaling factor which allows to pass from α to β or vice versa. Another meaningful example, is the following. Consider the $1-$form

$$\alpha = \sin(2\pi x)dy \in \Lambda^1(M).$$

This vanishes on the whole set $A = \{(0, y) : y \in \mathcal{S}^1)\} \cup \{(1/2, y) : y \in \mathcal{S}^1)\}$ but, computing its exterior derivative we get

$$d\alpha = 2\pi\cos(2\pi x)dx \wedge dy,$$

which does not vanish on A. Let's now take $\omega = dx \wedge dy$. At this point consider a vector field X preserving both ω and α. A simple choice is the field

$$X = c\partial_y,$$

in fact the following result holds:

$$\begin{aligned}\mathcal{L}_X\alpha &= \mathcal{L}_X(\sin(2\pi x))dy + \sin(2\pi x)d(\mathcal{L}_X(y)) = \mathcal{L}_X(\sin(2\pi x))dy + \sin(2\pi x)d(c) = \\ &= \mathcal{L}_X(\sin(2\pi x))dy = 0.\end{aligned}$$

Even if we already know that this field admits for example the symmetry field $Y = \partial_x$, let's show how the procedure given in the proposition allows to recover a similar relevant symmetry. We set $Y \in \mathfrak{X}(M)$ in such a way that $i_Y\omega = \alpha$, namely

$$Y^1 dy - Y^2 dx = \sin(2\pi x)dy,$$

so $Y = \sin(2\pi x)\partial_x$. This is a relevant symmetry for X since $X \wedge Y \neq 0$ and $[X, Y] = 0$. Moreover, we have found it with a systematic approach.

5.2.3 Third density independent condition

This third condition is based on the results presented in the book [9]. Consider a vector field $X \in \mathfrak{X}(\mathbb{T}^2)$, with no singular points, which reads

$$X = F_1(x,y)\frac{\partial}{\partial x} + F_2(x,y)\frac{\partial}{\partial y}.$$

Recall that these two functions are $1-$periodic in both their entries when seen as functions from $\mathbb{R}^2$ to $\mathbb{R}$. Moreover, assume there is an invariant volume form $\omega = p(x,y)dx \wedge dy$ with $L_X\omega = 0$.

Theorem 13. *There exists an infinitely differentiable change of coordinates*

$$u = u(x,y), \quad v = v(x,y)$$

conjugating the field X into

$$\tilde{X} = F(u,v)\frac{\partial}{\partial u} + \alpha F(u,v)\frac{\partial}{\partial v} \tag{5.4}$$

where $\alpha \in \mathbb{R}$ and F is a positive function.

The proof of this theorem can be found in [9]. To proceed, let's remark that a constant vector field admits a symmetry field. Hence, if we are able to conjugate the vector field $\tilde{X}$ to a constant field

$$\hat{X} = \lambda_1\frac{\partial}{\partial z} + \lambda_2\frac{\partial}{\partial t},\ \lambda_1, \lambda_2 \in \mathbb{R},$$

then we can consider one of the following symmetries:

- $Y = \partial/\partial z$ if $\lambda_2 \neq 0$ or
- $Z = \partial/\partial t$ if $\lambda_1 \neq 0$.

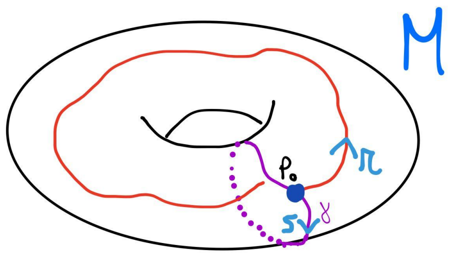


Figure 5.1: Construction of commuting fields with this closed curve γ

Indeed, when the vector field $\tilde{X}$ can be conjugated to $\hat{X}$, then X is not just integrable in the sense of Euler-Jacobi, but even B-integrable. A sufficient condition which guarantees we can do that, is the existence of a closed curve γ such that each of its points return to that curve at one and the same time. The existence of such a curve is discussed in [9] starting from page 94.

Supposing the existence of such a closed curve, we could eventually use it even in the proof of Euler-Jacobi instead of the arbitrary non-homotopically trivial closed loop we have built before. With this choice, we don't need to rescale the vector field $\tilde{X}$ because the parameter moving along its orbits is already a well set coordinate. Hence we have $[\tilde{X}, Y] = 0$ as in the proof of Euler-Jacobi since $\tilde{X} = \partial_s$, $Y = \partial_r$ are coordinate fields and they naturally commute.

5.2.4 Semi-global extension of these conditions

The aim of this section is to extend the constructions done on a single invariant torus to a tubular neighbourhood. Consider an orientable, connected and smooth $n-$dimensional manifold M and let V be a smooth vector field defined on it. Assume there is a submersion $F = (f_1, ..., f_{n-2}) : M \to \mathbb{R}^{n-2}$ whose components are first integrals of V and an invariant volume form $\mu = p d\boldsymbol{x}$. Assume V does not vanish on each invariant regular level set

$$N = \{m \in M : \, F(m) = \boldsymbol{c} \in \mathbb{R}^{n-2}\},$$

which is then diffeomorphic to $\mathbb{T}^2$. Restricting the vector field V and the form μ to N, we get exactly the local situation analyzed in the previous sections. Let's consider a point $m \in M$ with $F(m) = \boldsymbol{c}$. For a small-enough open neighbourhood $\mathcal{F} \subset \mathbb{R}^{n-2}$ of $\boldsymbol{c}$ we can define on $\mathcal{U}(N) = F^{-1}(\mathcal{F})$ the coordinate system

$$(f_1, ..., f_{n-2}, x, y) \in \mathbb{R}^{n-2} \times \mathbb{T}^2.$$

In these coordinates, the vector field V defining the dynamics takes the form

$$V|_{\mathcal{U}(N)} = V^1(f_1, ..., f_{n-2}, x, y)\frac{\partial}{\partial x} + V^2(f_1, ..., f_{n-2}, x, y)\frac{\partial}{\partial y}$$

since

$$df_i(V) = L_V f_i = 0 \quad \forall i = 1, ..., n-2.$$

When we consider its local restriction to N, we get

$$V|_N = X = V^1(\boldsymbol{c}, x, y)\frac{\partial}{\partial x} + V^2(\boldsymbol{c}, x, y)\frac{\partial}{\partial y}.$$

All the approaches presented in the previous section provide conditions under which this local field X admits a symmetry. The problem is if we can extend the analysis done up to now to a tubular neighbourhood $\mathcal{U}(N)$ of this torus. The last construction we have done, the third one with the closed curve γ, can be extended semi-globally exactly as for Euler-Jacobi Theorem. What actually changes from that Theorem, is just the closed curve we consider on the invariant torus.

To conclude this Chapter, let's analyze the semi-global version of the second condition. The smooth $1-$form $\alpha \in \Lambda^1(N)$ preserved by X can be smoothly extended to the whole $\mathcal{U}(N)$. Let

$$\alpha = a(\boldsymbol{c}, x, y)dx + b(\boldsymbol{c}, x, y)dy,$$

then we can extend it to $\mathcal{U}(N)$ as

$$\hat{\alpha} = \alpha \wedge df_1 \wedge ... \wedge df_{n-2} \in \Lambda^{n-1}(M).$$

This $(n-1)-$form is preserved even by the whole vector field $V \in \mathfrak{X}(\mathcal{U}(N))$. Indeed,

$$\mathcal{L}_V \hat{\alpha} = \mathcal{L}_V(\alpha \wedge df_1 \wedge ... \wedge df_{n-2}) = \mathcal{L}_V(\alpha) \wedge df_1 \wedge ... \wedge df_{n-2} = 0$$

since on each invariant level set we have $\mathcal{L}_X \alpha = 0$. Hence setting $Y \in \mathfrak{X}(\mathcal{U}(N))$ in such a way that $i_Y \mu = \hat{\alpha}$, guarantees:

$$i_{[V,Y]}\mu = \mathcal{L}_V(i_Y\mu) - i_Y(\mathcal{L}_V\mu) = \mathcal{L}_V(i_Y\mu) = \mathcal{L}_V\hat{\alpha} = 0.$$

This shows that when we are able to build the symmetry of the field X on N through the second approach, then we can follow the same idea to build a semi-global symmetry field and hence get broad-integrability hypotheses on $\mathcal{U}(N)$.

Appendix A

Poincaré-Hopf Theorem

In this Appendix, we briefly define the notions required to state the Poincaré-Hopf Theorem and then write its statement for a general compact and oriented manifold.

Definition 15 (Index of a vector field on $\mathbb{R}^n$)**.** Let $X \in \mathfrak{X}(U)$ be a vector field on $U \subset \mathbb{R}^n$. If $X(P) = 0$ for some $P \in U$, we can define the index of X at P, $ind_P(X)$, as the degree of $\psi : D \to S^{n-1}$ defined as $\psi(x) = X(x)/|X(x)|$, where $D \subset U$ is a sphere around P containing just the stationary point P of X.

This definition can be extended to a vector field X defined on a smooth manifold M of dimension n simply by choosing a local chart (U, φ). Indeed, if $\varphi : U \to \mathbb{R}^n$ is a smooth local chart, then we define

$$ind_P X := ind_{\varphi(P)}(\varphi_* X).$$

The index of a vector field on a smooth manifold is independent on the choice of the local chart.

Theorem 14 (Poincaré-Hopf)**.** *Consider a smooth, oriented and compact manifold M and a vector field $X \in \mathfrak{X}(M)$ with a finite number of critical points $P_1, ..., P_k \in M$. Then the sum of their indexes fully characterize the Euler characteristic of M, indeed*

$$\chi(M) = \sum_{i=1}^{k} ind_{P_i} X.$$

To conclude this Appendix, let's notice that this result holds even in the case of $2-$dimensional manifolds, i.e. the one we are interested in for the proof of Euler-Jacobi integrability Theorem. Moreover, connected $2-$dimensional manifolds (i.e. surfaces) admit a quite general classification which allows to recover the Proposition 22. Without presenting the whole classification, let's just highlight that smooth, orientable, connected and compact $2-$dimensional manifolds are homeomorphic either to $\mathcal{S}^2$ or to the connected sum of g $2-$dimensional tori, where g is the genus of M. Moreover, we know that

$$\chi(M) = 2 - 2g$$

and since, in Euler-Jacobi integrability Theorem, the considered vector field is supposed to be non-vanishing, follows that by Poincaré-Hopf Theorem $\chi(M) = 0$. This implies that $2g = 2$, i.e. $g = 1$. In particular, M is homeomorphic to $\mathbb{T}^2 \simeq \mathcal{S}^1 \times \mathcal{S}^1$.

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